A Characterization of Trivial Unawareness

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1. Introduction

The standard framework available within economics to formalize the concept of unawareness is that of state-space models. These structures are a basic tool of decision theory and have been traditionally used in the study of agents' knowledge and beliefs. Unfortunately, Dekel, Lipman and Rustichini [1998] have demonstrated that, within this framework, certain definitions of unawareness can only be captured in a trivial way.

In this work, the latter impossibility theorem is analyzed using *propositional models*. Such an analysis is useful because standard state-space structures can be derived from a particular class of propositional models: those satisfying *event-sufficiency* and the *real-states assumption*. As a more general setting, they allow a better understanding of the problems associated with modeling unawareness.

The main finding of this work is a characterization of trivial unawareness in terms of an axiom based on Fagin and Halpern's [1988] seminal paper on beliefs, awareness, and limited reasoning. The axiom, that is referred to as *unawareness generated by primitive propositions,* captures a very intuitive interpretation about unawareness without making any explicit connection to knowledge: an agent fails to conceive a relevant aspect of the world if and only if he is also failing to conceive a primitive element within that description.

The intuition behind the result is simple. A setting that dispenses the use of propositions and focuses solely on the worlds described –as standard state-space models and its "founding" class of propositional models– cannot be used to express a concept such as lack of conception, that intrinsically relies on the relationship between primitive and more complex propositions of a language. Thus, whenever unawareness is generated by primitive propositions, it can only be trivial: if the agent is unaware of a proposition in the language, he must be unaware of all of them.

The characterization is valid not only for the class of propositional models satisfying event sufficiency and the real-states assumption but also for a broader class of propositional models. Dekel, Lipman and Rustichini suggest that if distinction is to be made between the agent and the analyst's description of the world, the real-states assumption should be dropped. This idea is followed and instead of requiring that every proposition in the language be either true or false, a weaker restriction is proposed; namely, that "single-proposition tautologies" exist for every primitive proposition within the language.

To realize that the real-states assumption implies the single-proposition tautology restriction is not difficult at all. Under the real-states assumption, the assertion *p or not p,* that includes only one proposition, is true in every state of the world, and thus, is a single-proposition tautology. The converse does not necessarily hold. One can think of models where, for some states of the world, proposition p is neither true nor false and yet the statement *either agent knows p, does not know p, or is unaware of p* is true in all the states of the world. Thus, this model fulfills the single-proposition tautology restriction but violates the real-states assumption.

This work is organized as follows. In section 2, the propositional models setting is described along with event sufficiency and the real-states assumption. In section 3, Dekel, Lipman and Rustichini's impossibility result is demonstrated in terms of propositional models in order to understand the meaning of trivial unawareness within the new framework. In section 4, the characterization of trivial unawareness is proved, and its extension is discussed. Section 5, concludes.

2. Propositional models

2.1. Language and Semantic Structures

A propositional model is an ordered pair consisting of a *language* and a *semantic structure*. A language is defined in the following way. With a given set of agents $\{1, 2, ...n\}$, the descriptions that can be made about the world (formulas), are formed starting from a set $\Phi = \{p, q, ...\}$ of primitive propositions and then closing off under conjunction $\wedge$, negation $\neg$ and two modal operators for each agent: knowledge k_i and unawareness u_i. The formulas $p \vee q$ and $p \Rightarrow q$ constitute abbreviations of $\neg(\neg p \wedge \neg q)$ and $\neg p \vee q$ respectively. The resulting language is denoted as $L_n^{k,u}(\Phi)$. Intuitively, a language is nothing more than operations between a set of primitive propositions following some rules.

A *semantic structure* is a formal model that assigns a *truth value* to the propositions of a given language. Traditionally, the semantic structure of the language is constructed using a "states of the world" framework. The truth value of each formula is contingent on the state of the world considered.

Definition 1 *Let L be a language and A a finite subset of the natural numbers. Define a* ***semantic structure*** *over language L as a pair $M = (\Omega, T)$, where Ω is finite and $T : L \ x \ \Omega \rightarrow A \cup \{0, 1\}$ is a truth assignment for the elements of L.*

Each element $\omega \in \Omega$ is called a *state of the world* and Ω is referred to as the *state-space.* Intuitively, a state of the world is a set of descriptions based on the propositions of language L. The function $T : L \ x \ \Omega \rightarrow A \cup \{0,1\}$ contains the information concerning such descriptions. $T(\phi, \omega) = 1$ means the proposition ϕ is true in state ω and, correspondingly, $T(\phi, \omega) = 0$ means the proposition $\neg\phi$ is true in the state ω. The set A is used in the codomain of mapping T to allow for a more general truth assignment for propositions in the language. Therefore, whenever $T(\phi, \omega) \notin \{0,1\}$, neither proposition ϕ nor proposition $\neg\phi$ is true at state of the world ω.

Although the definition of semantic structure pretends to be as general as possible, some restrictions must be imposed to ensure congruence between the truth values of propositions. In particular, the following is required: $T(\phi, \omega) = 1 \Leftrightarrow T(\neg\phi, \omega) = 0$; and $T(\phi \wedge \psi, \omega) = 1 \Leftrightarrow T(\phi, \omega) = 1$ and $T(\psi, \omega) = 1$.

Too keep with the standard notation, the truth relation " $\models$ " induced by the semantic M is defined in the following way: $(M, \omega) \models \phi \Leftrightarrow T(\phi, \omega) = 1$. Where $(M, \omega) \models \phi$ means that under semantic structure M, proposition ϕ is true at the state of the world ω.

Now the fundamental tool used in this work is defined

Definition 2 *A* ***propositional model*** *is an ordered pair* $(L_n^{k,u}(\Phi), M)$, *in which* $L_n^{k,u}(\Phi)$ *denotes a language and* M *a semantic structure.*

For the sake of clearness –and following Dekel, Lipman and Rustichini– a full blown description of modal logic is avoided and propositional models are presented as simple as possible. For further reference go to Chellas [1980] and Fagin, Halpern, Moses and Vardi [1995].

2.2. *Event sufficiency and the real-states assumption*

It is not difficult to recast a propositional model as a state-space model. Let ϕ be a proposition of language $L_n^{k,u}(\Phi)$ in model $(L_n^{k,u}(\Phi), M)$ and consider the following subsets of the state-space:

$$E(\phi) = \{\omega \in \Omega \mid (M, \omega) \models \phi\},$$
$$K_i(E(\phi)) = \{\omega \in \Omega \mid (M, \omega) \models k_i\phi\}$$
$$U_i(E(\phi)) = \{\omega \in \Omega \mid (M, \omega) \models u_i\phi\}$$

$E(\phi)$ is the set of states of the world in which proposition ϕ is true. In terms of the state-space framework, $E(\phi)$ represents the *event* "ϕ is true". Accordingly, K_i and U_i , that are operators defined over events, represent the knowledge and unawareness of agent i about event $E(\phi)$.

These subsets of the state space allow the derivation of an *"event-based"* model from a *"propositional-based"* setting. However, note that in the latter definitions, the events are still dependent on the particular proposition chosen. Standard state-space structures require to dispense with this relation. This is precisely the role of event sufficiency and the real-states assumption.

Definition 3 *Let* $(L_n^{k,u}(\Phi), M)$ *be propositional model:*

1. *The propositional model satisfies* ***event sufficiency*** *if for any two propositions* $\phi, \psi \in L_n^{k,u}(\Phi)$ *satisfying the property* $(M, \omega) \models \phi \Leftrightarrow (M, \omega) \models \psi$, *then it is the case that* $(M, \omega) \models k_i\phi \Leftrightarrow (M, \omega) \models k_i\psi$ *and* $(M, \omega) \models u_i\phi \Leftrightarrow (M, \omega) \models u_i\psi$.

2. *The propositional model satisfies the* ***real-states assumption*** *if for every* $\phi \in L_n^{k,u}(\Phi)$ *it follows that* $(M, \omega) \not\models \phi \Rightarrow (M, \omega) \models \neg\phi$

Event sufficiency requires that the only relevant feature to talk about an agent's knowledge and unawareness is the set of states of the world in which a proposition is true. The proposition chosen is irrelevant, since for any two propositions corresponding to the same event –i.e., being true in exactly the same states of the world– the agent knows one if and only if he knows the other (the same for unawareness).

The real-states assumption imposes an important restriction on the truth value assignments of the semantic structure: if a proposition ϕ is not true at the state of the world ω, then at that state, the negation of that proposition is necessarily true. This restriction precludes models in which a given proposition is neither true nor false at a given state of the world.

Once these restrictions are imposed on propositional models, the standard state-space framework is obtained. Propositions are dispensed and the set of states where the agent knows or is unaware of something are identified without further reference to the propositional setting.

3. Dekel, Lipman and Rustichini's impossibility result

Dekel, Lipman and Rustichini's approach to unawareness is axiomatic. Although they use propositional models to identify the "founding" class of standard state-space structures, their result is phrased by using the event-based approach.

They impose some axioms on unawareness in terms of the knowledge operator, and they study which class of models can accommodate their definition of unawareness. Their main result is an impossibility theorem: in the class of standard state-space models, any unawareness operator that satisfies the axioms must be trivial. In this section this result is presented in terms of propositional models.

Consider propositional model $(L_n^{k,u}(\Phi), M)$. Dekel, Lipman and Rustichini's axioms are restrictions imposed on the semantic structure of the language. These restrictions are used to establish a connection between knowledge and unawareness.

Axiom 1 (Plausibility) *$(L_n^{k,u}(\Phi), M)$ is plausible if for any $\phi \in L_n^{k,u}(\Phi)$,it follows that $(M,\omega) \models u_i\phi \Longrightarrow (M,\omega) \models \neg k_i\phi \wedge \neg k_i \neg k_i \phi$.*

Axiom 2 (KU Introspection) *$(L_n^{k,u}(\Phi), M)$ satisfies KU introspection if for any $\phi \in L_n^{k,u}(\Phi)$, $(M,\omega) \models u_i\phi \Longrightarrow (M,\acute{\omega}) \not\models k_i u_i \phi \ \forall \ \acute{\omega} \in \Omega$.*

Axiom 3 (AU introspection) *$(L_n^{k,u}(\Phi), M)$ satisfies AU introspection if for any $\phi \in L_n^{k,u}(\Phi)$, $(M,\omega) \models u_i\phi \Longrightarrow (M,\omega) \models u_i u_i \phi$.*

Axiom 4 (Weak neccesitation) *$(L_n^{k,u}(\Phi), M)$ satisfies weak necessitation if for every tautology $\tau(\phi) \in L_n^{k,u}(\Phi)$ involving only proposition ϕ, $(M,\omega) \models \neg u_i\phi \Longrightarrow (M,\omega) \models k_i \tau(\phi)$.*

Except for weak necessitation, all axioms are simple restatements of their counterparts for state-space models. In the case of weak necessitation, Dekel, Lipman and Rustichini refer to proposition $\tau(\phi) \in L_n^{k,u}(\Phi)$ not as a tautology that involves only proposition ϕ, but rather as an "obvious" tautology. Since they do not explicitly define what is the meaning of "obvious", the term is taken here as a proposition that is true in all the states of the world (standard definition of tautology) and that involves only one proposition.

To better understand Dekel, Lipman and Rustichini's result, the following Lemma is proved. Under the real-states assumption, the first three axioms imply a strong relationship between unawareness and knowledge: if agent is unaware of a proposition, he must not know at least one tautology in the language.

Lemma 1 *Consider the class of propositional models* $(L_n^{k,u}(\Phi), M)$ *that satisfies the real-states assumption. If the axioms of plausibility, KU introspection, and AU introspection hold, then:*

$$(M,\omega) \models u_i\phi \Longrightarrow (M,\omega) \models \neg k_i\tau \text{ for some tautology } \tau \in L_n^{k,u}(\Phi)$$

Proof. By the axiom of KU introspection follows that $(M,\omega) \models u_i\phi \Longrightarrow (M,\acute{\omega}) \not\models k_iu_i\phi \ \forall \ \acute{\omega} \in \Omega$. By the real-states assumption, $(M,\acute{\omega}) \not\models k_iu_i\phi \Longrightarrow (M,\acute{\omega}) \models \neg k_iu_i\phi \ \forall \ \acute{\omega} \in \Omega$. Hence, $\neg k_iu_i\phi$ is a tautology. From Plausibility and AU introspection $(M,\omega) \models u_i\phi \Longrightarrow (M,\omega) \models u_iu_i\phi \Longrightarrow (M,\omega) \models \neg k_iu_i\phi \wedge \neg k_i\neg k_iu_i\phi$. Thus, agent does not know tautology $\tau \equiv \neg k_iu_i\phi$. ■

Note that when Lemma 1 is combined with event sufficiency, unawareness of a proposition implies the agent does not know any tautology in the language. In addition, the contrapositive of the weak necessitation axiom allows the following conclusion: not knowing a single proposition tautology implies unawareness of such proposition. These two facts are the main elements of the impossibility result.

Theorem 1 (Dekel, Lipman and Rustichini) *Consider the class of propositional models that satisfies event sufficiency and the real-states assumption. If plausibility, KU introspection, AU introspection, and weak necessitation hold, then:*

$$(M,\omega) \models u_i\phi \Longrightarrow (M,\omega) \models u_i\psi \ \forall \ \psi \in L_n^{k,u}(\Phi)$$

Proof. From Lemma 1 and event sufficiency, $(M,\omega) \models u_i\phi \Longrightarrow (M,\omega) \models \neg k_i\tau$ for every tautology $\tau \in L_n^{k,u}(\Phi)$. Take an arbitrary $\psi \in L_n^{k,u}(\Phi)$. Under the real-states assumption $\tau^*(\psi) \equiv \psi \vee \neg\psi$ is a tautology that contains only one proposition. With the contrapositive of weak necessitation and the real-states assumption, $(M,\omega) \models \neg k_i\tau^*(\psi) \Longrightarrow (M,\omega) \models u_i\psi$ for $\psi \in L_n^{k,u}(\Phi)$. Since ψ is arbitrary, then $(M,\omega) \models u_i\phi \Longrightarrow (M,\omega) \models \neg k_i\tau^*(\psi) \Longrightarrow (M,\omega) \models u_i\psi \ \forall \ \psi \in L_n^{k,u}(\Phi)$. ■

Understanding trivial unawareness is important. In terms of the state-space framework, triviality does not mean that the subsets of the state-space where agent is unaware is either the null set or the whole space; triviality means that for any two events, the subset of the state space in which agent is unaware is the same.

In the propositional setting trivial unawareness means that if at some state of the world the agent is unaware of a proposition ϕ, he is unaware of all propositions in the language. Formally:

$$(M, \omega) \models u_i\phi \Longrightarrow (M, \omega) \models u_i\psi \ \forall \ \psi \in L_n^{k,u}(\Phi).$$

In the following section, a characterization of trivial unawareness is presented.

4. A characterization of trivial unawareness

Fagin and Halpern [1988] suggested some restrictions on the definition of *awareness* in order to capture the idea of lack of conception. One of these restrictions was that at each state of the world ω, there existed a set of primitive propositions $\Phi(\omega)$ such that agent was aware of exactly those constructed formulas constructed that use primitive propositions belonging to $\Phi(\omega)$.

Following this definition, the axiom of *unawareness generated by primitive propositions* is proposed: an agent is unaware of a proposition $\phi \in L_n^{k,u}(\Phi)$ if and only if there is a primitive proposition p –used in the construction of ϕ– of which agent is unaware. This axiom imposes a natural restriction on unawareness without making any explicit connection to knowledge.

The main conclusion of the paper is the following: inside the class of propositional models that satisfies event sufficiency and the real-states assumption, the definitions of *trivial unawareness* and *unawareness generated by primitive propo-*

sitions are equivalent (it is not difficult to notice that one side of the contention is immediate)

Definition 4 (Unawareness generated by primitive propositions) *Let the mapping* $\Gamma : L_n^{k,u}(\Phi) \to \Phi$ *be a correspondence that associates to every proposition* $\phi \in L_n^{k,u}(\Phi)$ *the set* $\Gamma(\psi) \subseteq \Phi$ *that contains all primitive propositions involved in proposition* ϕ*. The propositional model* $(L_n^{k,u}(\Phi), M)$ *satisfies the unawareness generated by primitive propositions axiom if for every* $\phi \in L_n^{k,u}(\Phi)$ *:*

$$(M,\omega) \models u_i\phi \Leftrightarrow \exists \text{ a primitive proposition } p \in \Gamma(\phi) \text{ such that } (M,\omega) \models u_i p.$$

Proposition 1 *Consider the class of propositional models that satisfies event sufficiency and the real-states assumption. Unawareness is trivial if and only if unawareness is generated by primitive propositions.*

Proof. By definition of trivial unawareness $(M,\omega) \models u_i\phi \implies (M,\omega) \models u_i\psi$ for any proposition $\psi \in L_n^{k,u}(\Phi)$. In particular $(M,\omega) \models u_i p$ for every primitive proposition $p \in \Gamma(\phi)$. In the same way, if there exists a primitive proposition p satisfying $(M,\omega) \models u_i p$, then $(M,\omega) \models u_i\phi$, for any ϕ such that $p \in \Gamma(\phi)$. So, $(M,\omega) \models u_i\phi \Leftrightarrow \exists$ a primitive proposition $p \in \Gamma(\phi)$ such that $(M,\omega) \models u_i p$. Thus, unawareness is generated by primitive propositions.

For proving the converse, if unawareness is generated by primitive propositions $(M,\omega) \models u_i\phi$ implies the existence of a primitive proposition $p \in \Gamma(\phi) \subseteq \Phi$ such that $(M,\omega) \models u_i p$. Define $\tau := p \vee \neg p$. Note that under the real-states assumption, τ is a tautology. In addition, $\Gamma(\tau) = p$. Once again by unawareness generated by primitive propositions, it follows $(M,\omega) \models u_i\tau$. Now take any other primitive proposition $q \in \Phi$ with $p \neq q$. Define $\tau' := q \vee \neg q$. Since τ' is also a tautology, $(M,\omega) \models \tau \Leftrightarrow (M,\omega) \models \tau'$ and by event sufficiency $(M,\omega) \models u_i\tau \Leftrightarrow$

$(M, \omega) \models u_i\tau'$. Since $\Gamma(\tau') = q$, unawareness generated by primitive propositions allows the conclusion $(M, \omega) \models u_i\tau' \Rightarrow (M, \omega) \models u_i q$. So, $(M, \omega) \models u_i\phi \Rightarrow (M, \omega) \models u_i\tau \Rightarrow (M, \omega) \models u_i\tau' \Rightarrow (M, \omega) \models u_i q$. Since q was taken arbitrarily $(M, \omega) \models u_i\phi \Rightarrow (M, \omega) \models u_i q \; \forall \; q \in \Phi$, that implies $(M, \omega) \models u_i\psi \; \forall \; \psi \in L_n^{k,u}(\Phi)$. ■

Although event sufficiency is fundamental to the proof of Proposition 1, the real-states assumption is only required to associate a single-proposition tautology to every primitive proposition in the language. Thus, the characterization result can easily be extended to the class of propositional models satisfying this *"single-proposition tautology requirement",* that as mentioned in section 1, is broader than the class that satisfies the real-states assumption.

The proof remains practically the same. The only difference is that a single-proposition tautology for every primitive proposition is assumed to exist without being explicitly constructed. The extension is important since it includes propositional models in which unawareness is introduced as a third truth value of the semantic structure. For example, if at a given state of the world a proposition ϕ could be either true, false, or otherwise, agent would be unaware of it, then the proposition $\tau : p \vee \neg p \vee u_i p$ would be a single-proposition tautology.

5. Conclusion

The characterization result of Proposition 1 simplifies understanding the aspects of unawareness that make it incompatible with event sufficiency and the real-states assumption. As it has been shown, the problems with modelling unawareness arise not from the restrictions imposed on its meaning in terms of lack of knowledge, but rather from the restrictions imposed on its structure.

Intuitively, the idea of agents' lack of conception involves a relationship between primitive and more complex propositions of the language. A framework in which the set of the states of the world becomes the only relevant aspect for constructing knowledge and unawareness –as required by event sufficiency– cannot express such a relationship.

Dekel, Lipman and Rustichini's result uses the plausibility, KU introspection and AU introspection axioms, along with the real-states assumption to establish the following fact: unawareness of a proposition implies that the agent does not know an specific tautology of the language (Lemma 1). Event sufficiency and weak necessitation turn this lack of knowledge into unawareness of all propositions.

A similar structure is required to prove that *unawareness generated by primitive propositions* implies *trivial unawareness* (for the other side of the contention, neither event sufficiency nor the real-states assumption are necessary). First, unawareness of an arbitrary proposition is converted into unawareness of a tautology. The latter, combined with event sufficiency and the real-states assumption, allows the conclusion that unawareness must be trivial.

The characterization result and its extension show the extent to which the real-states assumption has to be relaxed in order to capture non-trivial notions of unawareness while maintaining the relationship between unawareness of primitive and more complex propositions in the language. This is of particular importance, and suggests that a necessary condition to capture the *unawareness generated by primitive propositions axiom* in an event sufficiency setting, is the incapability of associating single-propositions tautologies to every primitive proposition in the language.

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