Lower bounds and recursive methods for the problem of adjudicating conflicting claims*

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Abstract

For the problem of adjudicating conflicting claims, we study lower bounds on the awards of each agent. We propose extending a lower bound by performing the following operation: (i) for each problem, assign the lower bound and revise the problem accordingly; (ii) assign the bound of the revised problem. The "recursive-extension" of a bound is the lower bound obtained by recursive application of this operation. We provide necessary and sufficient conditions on a lower bound for there to be a unique rule satisfying its recursive-extension. We show that, under these conditions, the rule satisfying the recursive-extension of a bound is the unique rule satisfying the following property: the awards assigned by a rule should be obtainable in two ways: (i) directly applying the rule to the problem or (ii) first assigning the bound and revising the problem accordingly, and then applying the rule to the revised problem. Then, we study whether this procedure leads to well-behaved lower bounds.

Keywords: Claims problems, Lower bounds, Recursive methods.

JEL Classification Numbers: C79, D63, D74

1 Introduction

When a group of agents have claims over some resource but there is not enough of it to honor all of the claims, how should the resource be divided among them? When the resource consist of a positive amount of a single infinitely divisible good, we have a "claims problem". A "rule" assigns, for each problem, a division of the resource among the agents. In this paper we study lower bounds on how much of the resource each agent should receive. A "lower bound" assigns, for each problem and each agent, a minimal amount of the resource that she should receive. A rule satisfies a lower bound if, for each problem and each agent, the rule assigns her at least as much as the lower bound does. Several lower bounds have been proposed in the literature. Rules satisfying them (along with other properties) have been identified, and some bounds have appeared in characterizations of rules.

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In economics, lower bounds are imposed for many different reasons: Participation constraints (i.e. individual rationality) are widespread; they are imposed when designing mechanisms and agents cannot be forced to participate. In other settings, lower bounds are used as fairness criteria; when dividing a social endowment of goods it is often thought desirable to provide to each agent a welfare level that is at least as large as her welfare level obtained from dividing the endowment equally among the agents. For our model both interpretations are possible: For example, in a bankruptcy situation, where the value of a firm has to be divided among a set of creditors, a lower bound provides insurance to the creditors. In estate division problems, a lower bound can be interpreted as a partial division of the resource that all agents agree upon from fairness considerations, but there may still be some conflict over the remaining resources.

An alternative, more technical, interpretation of a lower bound is as an inefficient rule. A lower bound assigns for each problem a division of the resource, but fails to allocate it entirely. Under this interpretation, our results provide conditions under which extending an inefficient rule leads to an efficient one.

Recently, a lower bound was introduced which assigns to each agent the minimum of (i) her claim divided by the number of agents, and (ii) the amount to divide equally shared by all the agents (Moreno-Ternero and Villar 2004). Many rules satisfy this bound, but only one satisfies the following invariance property: the awards assigned by the rule should be obtainable in two ways: (i) applying the rule directly or (ii) first assigning the lower bound, then applying the rule to an appropriately revised problem (Dominguez and Thomson 2006).

In this paper we undertake a general study of this type of invariance property. For some lower bounds the invariance property singles out a rule, while for other lower bounds it does not. We show that a sufficient condition on a lower bound for the invariance property to yield a unique rule is that the bound assigns, for each problem with a positive endowment and to each agent with a positive claim, a positive amount. Moreover, the weaker condition that the bound assigns, for each problem in which at least one agent has a positive claim, a positive amount to at least one agent is not only sufficient but also necessary for a unique rule to emerge.[1] This is our main result. It also provides a new interpretation of the invariance property: if, after assigning to each agent the amount assigned to her by the lower bound and revising the problem accordingly, the lower bound of the revised problem still assigns positive amounts to some agents, then we should apply it again. The "recursive-extension of a bound" is obtained by recursively revising the problem and extending the bound accordingly. Whenever the recursive-extension of a bound singles out a rule, it is the unique rule satisfying the invariance property with respect to the bound.

We then study the behavior of the recursive-extension of a bound. To do so, we introduce some properties of good behavior for rules and lower bounds.[2] Most of the properties that have been formulated for rules can also be directly applied to lower bounds; their interpretation remains valid and the properties retain their appeal. Some properties, when imposed on a bound, are automatically satisfied by its recursive-extension, and we say that the property is "inherited". In order to study the behavior of the recursive-extension of a

[1] Since lower bounds are interpreted as providing protection to the agents, the condition that a lower bound assigns a positive amount to each agent seems desirable; for the alternative interpretation of a lower bound as an inefficient rule, the condition that it assigns a positive amount to at least one agent can be seen as weakening the efficiency requirement, but not as much as to allow a rule to distribute none of the resource.

[2] Most of the literature on claims problems has been axiomatic. For a discussion on the axiomatic method, applied to this problem and other economic problems, see Thomson (2001).

bound we undertake a systematic investigation of inheritance. Some properties are inherited on their own, while others are inherited only when imposed together with other properties.

2 The model

There is a social **endowment** $E \in \mathbb{R}_+$ of an infinitely divisible good.[3] The endowment has to be divided among a set $N = \{1, ..., n\}$ of **agents**.[4] Each agent $i \in N$ has a **claim** $c_i \in \mathbb{R}_+$ on the endowment. The endowment is not sufficient to honor all the claims. A **claims problem** (or simply a problem) is a vector $(c, E) \in \mathbb{R}_+^{|N|} \times \mathbb{R}_+$ such that $E \leq \sum_{i \in N} c_i$. Let $\mathcal{C}^N$ be the set of all problems with population N.[5]

An **awards vector**, $x \in \mathbb{R}_+^n$, for the problem $(c, E) \in \mathcal{C}^N$ distributes the resources among the agents. It assigns to each agent $i \in N$ the amount x_i of the resource. For each problem $(c, E) \in \mathcal{C}^N$ we restrict attention to awards vectors which assign, to each agent, a non-negative amount of the resource no larger than her claim, and which distribute the endowment entirely, that is:

(i) $0 \leqq x \leqq c$,[6]

(ii) $\sum_{i \in N} x_i = E$.

Let $X(c, E)$ denote the **set of awards vectors** for (c, E). Condition (i) are the **non-negativity** and **claims boundedness** restrictions. Condition (ii) is **efficiency**. The set of vectors satisfying *efficiency* is denoted $F(c, E)$. For some problems, the set of awards vectors consists of a single element.[7] Such problems are called **degenerate**, and we obtain the awards vector by definition of a rule. Whenever the set of awards vectors contains more than one element, the problem is **non-degenerate**.

Definition 1. A **rule** is a function $\varphi : \mathcal{C}^N \to \mathbb{R}^n$, which maps each problem $(c, E) \in \mathcal{C}^N$, to an awards vector $\varphi(c, E) \in X(c, E)$.

A graphical representation of a rule is by means of its **paths of awards** (see Figure 1). Given a rule φ, for each claims vector c, the path of awards of φ for c is the image of the function $\varphi(c, \cdot) : [0, \sum_i c_i] \to X(c, \cdot)$, which maps each endowment E, with $0 \leq E \leq \sum_i c_i$, into the awards vector assigned by the rule. It describes the path followed by the awards vectors when the endowment varies from 0 to the sum of the claims.

A **rights vector** $x \in \mathbb{R}_+^n$ assigns to each agent $i \in N$ the minimal guarantee x_i of the resource. For each problem $(c, E) \in \mathcal{C}^N$, we restrict attention to rights vectors that are bounded above by the claims vector, and are feasible, that is:

(i) $0 \leqq x \leqq c$,

(ii) $\sum_{i \in N} x_i \leq E$.

The **set of rights vectors** for the problem (c, E), denoted $Y(c, E)$, is the set of vectors satisfying conditions (i) and (ii). Condition (ii) is **feasibility**.

[3] The set $\mathbb{R}_+$ denotes the non-negative reals.

[4] We refer to the set of agents as the population of a problem.

[5] This model was first introduced by O'Neill (1982).

[6] Vector inequalities: $x \geqq y \Leftrightarrow$ for each $i \in N$, $x_i \geq y_i$; $x \geq y \Leftrightarrow x \geqq y$ and $x \neq y$; $x > y \Leftrightarrow$ for each $i \in N$, $x_i > y_i$.

[7] Such problems arise when $E = 0$ or $E = \sum_{i \in N} c_i$, or when all but one of the claims are equal to 0.

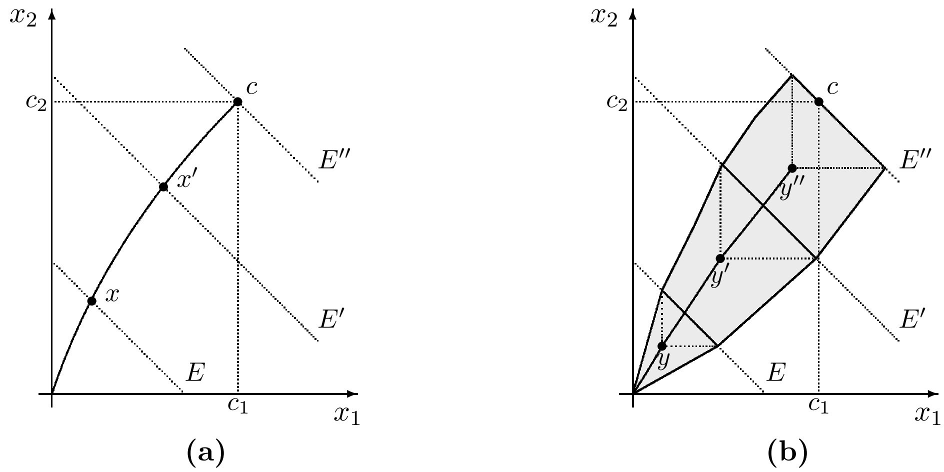

Figure 1: **Paths of awards and paths of acceptable vectors.** (a) Path of awards of φ for c. The path traced out by the awards vectors assigned by the rule as the endowment varies from 0 to the sum of the claims; the awards vectors $\varphi(c, E)$ and $\varphi(c, E')$ are denoted x and x' respectively. (b) The shaded area is the path of acceptable vectors of b for c; the rights vectors $b(c, E)$, $b(c, E')$, and $b(c, E'')$ are denoted y, y', and y'' respectively. The path traced by the lower bound vectors as the endowment varies from 0 to the sum of the claims passes through y, y' and y''.

Definition 2. A **lower bound** is a function $b : \mathcal{C}^N \to \mathbb{R}^n$, which maps each problem $(c, E) \in \mathcal{C}^N$, to a rights vector $b(c, E) \in Y(c, E)$

A graphical representation for a lower bound is by means of its **paths of acceptable vectors** (see Figure 1). Given a lower bound b, for each claims vector c, the path of acceptable vectors of b for c is the image of the correspondence $b(c, \cdot) : [0, \sum_i c_i] \to F(c, \cdot)$, which maps each endowment E with $0 \leq E \leq \sum_i c_i$, into the set of *efficient* vectors dominating the rights vector assigned by the lower bound. It is a set of vectors satisfying n restrictions. Each restriction assigns to one of the agents her right, and allows any division of the remainder among the other agents.

We say that a rule **satisfies a lower bound** if, for each problem, the awards vector assigned by the rule weakly dominates the rights vector assigned by the lower bound. Otherwise it **fails the lower bound**. Graphically, there is a simple procedure to determine whether or not a rule satisfies a lower bound. It is to verify whether, for each claims vector, the path of awards of the rule lies within the path of acceptable vectors of the lower bound (see Figure 1).

3 An inventory of lower bounds.

In this section, we introduce some existing lower bounds and formulate a new one. These lower bounds will help to illustrate our results. It is worth noting that the definition of a rule includes the requirement that, for each problem, awards are non-negative. Formally, this restriction can be seen as a lower bound. It is a basic property that all rules should satisfy, and following the literature, we embedded it into the definition of a rule. For the same reason we also impose it on lower bounds.

The first lower bound assigns to each agent the difference between the endowment and the sum of the claims of the other agents, or zero if this difference is negative. For each

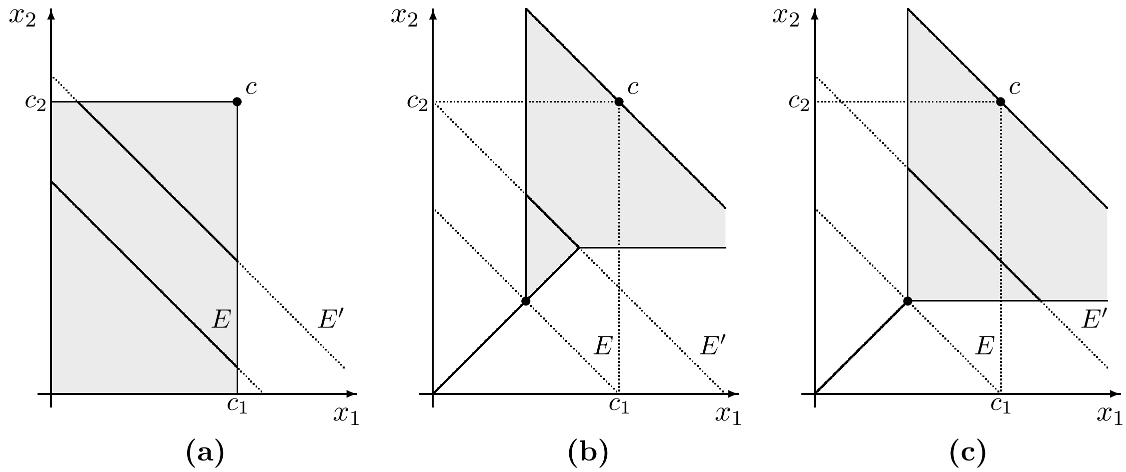

Figure 2: **Graphical representations of some lower bounds for $\mathbf{n = 2}$** (under the assumption that $c_1 < c_2$). (a) *Minimal rights.* The first restriction traces a path which follows the vertical axis until the point $(0, c_2)$, then follows a horizontal line until the claims vector; the second restriction traces a path which follows the horizontal axis until the point $(c_1, 0)$, then follows a vertical line until the claims vector. For each rule, its paths of awards lie inside the paths of acceptable vectors of *minimal rights*, hence all rules satisfy the bound. (b) *Reasonable lower bound.* The first restriction traces a path which follows the 45° line until the point $(\frac{c_1}{2}, \frac{c_1}{2})$, then follows a vertical line until the point $(\frac{c_1}{2}, c_2 + \frac{c_1}{2})$; the second restriction traces a path which follows the 45° line until the point $(\frac{c_2}{2}, \frac{c_2}{2})$, then follows a horizontal line until the point $(c_1 + \frac{c_2}{2}, \frac{c_2}{2})$. If $c_1 \neq c_2$, the paths of awards (for c) of the *proportional* and the *constrained equal losses* rules lie outside the path of acceptable vectors (for c) of *reasonable lower bound*, hence they fail the bound. (c) *Min lower bound.* Both restrictions trace out a path which follows the 45° line until the point $(\frac{\min\{c_i\}}{2}, \frac{\min\{c_i\}}{2})$, then the first restriction traces a vertical line until the point $(\frac{\min\{c_i\}}{2}, c_1 + c_2 - \frac{\min\{c_i\}}{2})$, the second traces the symmetric horizontal line.

$(c, E) \in \mathcal{C}^N$, agent i's **minimal right** is $m_i(c, E) = \max\{E - \sum_{j \in N \setminus i} c_j, 0\}$. The **minimal rights lower bound** (Curiel, Maschler, and Tijs 1987) is given by:

$$m(c, E) = (m_i(c, E))_{i \in N}.$$

Minimal rights is a very weak bound (see Figure 2). It is an implication of *non-negativity*, *claim-boundedness*, and *efficiency*. Thus, all rules satisfy *minimal rights*.

Suppose we want to guarantee to each agent a fixed proportion λ of her claim. Since the sum of the claims can be arbitrarily large relative to the endowment, the only feasible proportion is $\lambda = 0$. If we consider guaranteeing to each agent a fixed proportion of her claim truncated at the endowment there are many feasible proportions, and $\lambda = \frac{1}{n}$, is the highest such proportion. This observation provides the motivation of our next lower bound:

For each $(c, E) \in \mathcal{C}^N$, agent i's truncated claim is $t_i(c, E) = \min\{c_i, E\}$. The vector of **truncated claims** is $t(c, E) \equiv (t_i(c, E))_{i \in N}$. The **reasonable lower bound** (Moreno-Ternero and Villar 2004) is given by:

$$r(c, E) = \frac{1}{n} t(c, E).$$

Most rules in the literature satisfy the *reasonable lower bound.* Examples are the *constrained equal awards rule* (which selects $x \in X(c, E)$ such that for some $\lambda \in \mathbb{R}_+$, $x = (\min\{c_i, \lambda\})_{i \in N}$); the *random arrival rule* (which selects the average of the awards vectors obtained by imagining agents arriving one at a time and fully compensating them until the endowment runs out, under the assumption that all orders are equally likely); and the *Talmud rule* (which selects $x \in X(c, E)$ such that $x = (min\{\frac{c_i}{2}, \lambda\})_{i \in N}$ if $E \leq \frac{\sum c_i}{2}$, and $x = \frac{c}{2} + (\max\{\frac{c_i}{2} - \lambda, 0\})_{i \in N}$ otherwise).[8] However, the *proportional* rule (which selects $x \in X(c, E)$ such that for some $\lambda \in \mathbb{R}_+$, $x = \lambda c$) and the *constrained equal losses* rule (which selects $x \in X(c, E)$ such that for some $\lambda \in \mathbb{R}_+$, $x = (\max\{c_i - \lambda, 0\})_{i \in N}$) fail the *reasonable lower bound* (see Figure 2).

The next bound is new to the literature. To motivate it, start from a problem $(c, E) \in \mathcal{C}^N$ and consider a reference situation where each agent's claim is equal to the smallest claim in (c, E).[9] In such situation, agents should be treated equally; if the endowment is smaller than the common claim, equal division should prevail; if the endowment is greater than the common claim, each agent should receive at least $\frac{1}{n}$th of the common claim. In the problem (c, E), each agent's claim is at least as large as in the reference situation, hence her bound should not decrease. For each $(c, E) \in \mathcal{C}^N$, agent i's min lower bound is $\mu_i(c, E) = \frac{1}{n} \min\{\{c_j\}_{j \in N}, E\}$. The **min lower bound** is given by:

$$\mu(c, E) = (\mu_i(c, E))_{i \in N}.$$

Now, we note two ways to determine which of two lower bounds is stronger. Each defines a partial order on the space of rules. First, we compare, problem-by-problem, the rights vectors of the lower bounds. If for each problem, the rights vector of one lower bound dominates the rights vector of the other, then the former is stronger. Using this comparison, *minimal rights* is neither weaker nor stronger than either the *reasonable lower bound* or the *min lower bound*, but the *min lower bound* is weaker than the *reasonable lower bound*. Graphically, this comparison corresponds to containment of the paths of acceptable vectors (see Figure 3 (a)).

Second, we compare the sets of rules satisfying each of the two bounds. If the set of rules satisfying one lower bound is contained in the set of rules satisfying the other, then the former is stronger. Using this comparison, *minimal rights* is weaker than the *min lower bound*, which is weaker than the *reasonable lower bound*.[10] In fact, *minimal rights* is vacuously satisfied by all rules. Graphically, this comparison corresponds to containment of the paths of acceptable vectors restricted to the set of awards vectors. Given a lower bound, its paths of acceptable vectors restricted to the set of awards vectors are its **paths of acceptable awards** (see Figure 3 (b)).

3.1 Positivity, Strong Positivity, and conditional efficiency.

Now, we introduce some conditions on lower bounds. The first two conditions imply that the total amount assigned by the lower bound is positive:

Positivity: For each *non-degenerate problem* $(c, E) \in \mathcal{C}^N$, $b(c, E) \geq 0$.[11]

Strong positivity: For each *non-degenerate problem* $(c, E) \in \mathcal{C}^N$, each $i \in N$, if $c_i > 0$ then $b_i(c, E) > 0$.

[8] Dominguez and Thomson (2006) provide some general ways of identifying rules satisfying the *reasonable lower bound*.

[9] The reference situation may not be a problem since the sum of the claims may be smaller than the endowment. Reference to this situation can help determine the rights in the original problem.

[10] Even though the *min lower bound* is weaker than the *reasonable lower bound*, the *proportional* rule and the *constrained equal losses* rule violate the *min lower bound* (see Figure 2).

[11] Recall that the vector inequality $x \geq 0$ implies that: (i) for each $i \in N$ $x_i \geq 0$, and (ii) $x \neq 0$.

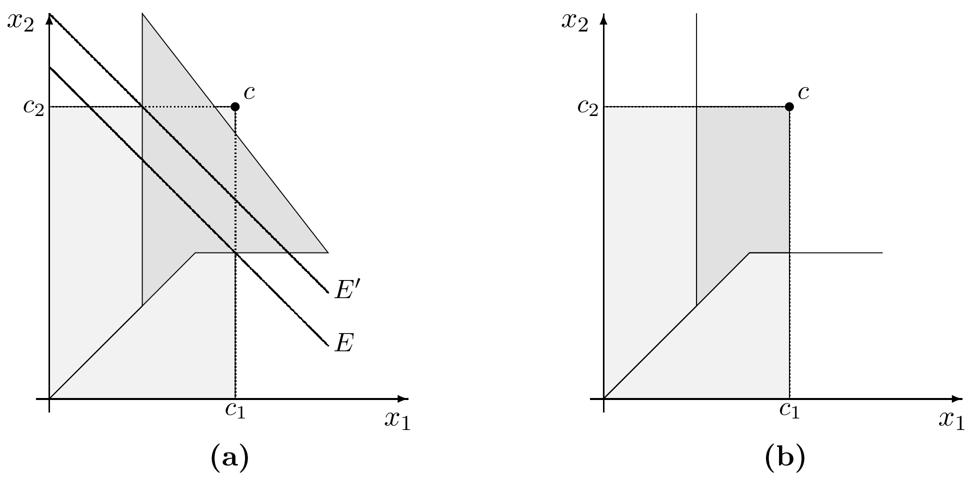

Figure 3: **Comparing minimal rights to the reasonable lower bound.** (a) *Problem-by-problem comparison.* The bounds are non-comparable. For endowments smaller than E, the rights vectors of the *reasonable lower bound* dominate the rights vectors of *minimal rights*; for endowments larger than E and smaller than E', there is no domination; for endowments greater than E', the rights vectors of *minimal rights* dominate the rights vectors of the *reasonable lower bound.* (b) *Set of rules satisfying each bound.* The paths of acceptable awards of the *reasonable lower bound* are contained in the paths of acceptable awards of *minimal rights.* Hence, the *reasonable lower bound* is stronger than *minimal rights.*

It is clear that *strong positivity* implies *positivity.* Both properties can be interpreted as protecting agents from receiving nothing. The first protects the agents as a group, the second protects each agent separately. Under this interpretation *strong positivity* is of particular interest. If we interpret lower bounds as *inefficient* rules the properties can be thought of as minimal *efficiency* requirements: a lower bound should assign a positive amount whenever possible.[12] It is easy to see that *minimal rights* and the *min lower bound* fail *positivity* (and hence *strong positivity*). The *reasonable lower bound* satisfies *strong positivity.* In two-agent problems so does the *min lower bound.*

When a problem is *degenerate*, there is no conflict among the agents. A lower bound should distribute the endowment fully:

Conditional efficiency: For each *degenerate problem* $(c, E) \in \mathcal{C}^N$, $b(c, E) = X(c, E)$.

Suppose we assign to each agent her right and revise the claims and the endowment accordingly. Given a rights vector x, for each $(c, E) \in \mathcal{C}^N$, the **x-partially resolved problem** is $p^x(c, E) = \big(((c - x)), E - \sum_{i \in N} x_i\big)$.[13] It defines the problem (c', E') where (i) for each agent $i \in N$, the claim c'_i is what remains of her initial claim after she is assigned her right and (ii) E' is the amount of the endowment remaining after assigning their rights to the agents. Given a lower bound b, For each $(c, E) \in \mathcal{C}^N$, we abuse notation and denote the $b(c, E)$-partially resolved problem, $p^b(c, E) \equiv p^{b(c,E)}(c, E)$.

[12]For this interpretation to be valid, we should require that the lower bound assigns a positive amount to each problem with a positive endowment.

[13]It is easy to see that the set of problems is closed under this operation.

4 Extending a lower bound

Suppose we agree on a lower bound and find that, for some problem, the rights vector of its *partially resolved problem* assigns a positive amount to at least one agent. In such situations we can extend the lower bound by adding to the rights vector of the original problem the rights vector of the revised problem. The *recursive-extension of a bound* is obtained by repeatedly performing this operation.

Similarly, starting from a rule and a lower bound, we can construct a new rule in the following way: for each problem, first assign to each agent her right, then apply the rule to the revised problem.[14] For some rule-bound pairs the constructed rule coincides with the original rule. We say that the rule is "*invariant under the assignment of the bound*".

In this section we show that, if a lower bound satisfies *positivity*, repeatedly assigning the lower bound singles out the unique rule satisfying *invariance under the assignment of the bound*. First, we formally define the recursive-extension of a lower bound, and illustrate the process using the lower bounds in Section 3. Then, we define *invariance under the assignment of a bound*. Finally, we state and prove our main theorem. We restrict attention to continuous lower bounds. Similar results can be obtained for discontinuous bounds, but the definition of the recursive-extension and the proofs are significantly easier for continuous bounds.

4.1 Recursive-extension

Consider a lower bound and extend it according to the operation described above. Then, extend the new lower bound using the same operation. The recursive-extension of the bound is obtained by repeatedly extending a lower bound.

Definition 3. Given a lower bound b, let $b^0 \equiv b$. For each $(c, E) \in \mathcal{C}^N$, and each $k \in \mathbb{N}$, the **k-extension of $\boldsymbol{b}$** is the lower bound defined by:

$$b^k(c, E) = b^{k-1}(c, E) + b^{k-1}(p^{b^{k-1}}(c, E)).$$

It is clear that for each each $(c, E) \in \mathcal{C}^N$, the sequence $\{b^k(c, E)\}$ is increasing and bounded. Thus, it has a limit, and this limit defines the rights vector assigned by the recursive-extension of the bound to the problem (c, E).

Definition 4. Given a lower bound b, for each $(c, E) \in \mathcal{C}^N$, the **recursive-extension of $\boldsymbol{b}$** assigns the rights vector $B(c, E)$, where:

$$B(c, E) = \lim_{k \to \infty} b^k(c, E).$$

If a lower bound is continuous, it is easy to see that its recursive-extension is also continuous.[15] The next lemma provides an alternative way of obtaining the recursive-extension of a lower bound.

[14] It is easy to see that this operation defines a rule. For a detailed study of this operation, with respect to *minimal rights*, see Thomson and Yeh (2005).

[15] To prove this claim it suffices to define the operator $T : \mathcal{B} \to \mathcal{B}$ which maps each lower bound into its 2-extension. Then, show that recursive application of the operator defines a Cauchy sequence in the space of continuous and bounded functions endowed with the sup-norm metric. Finally, note that this space is complete, hence it contains all it's limit points.

Lemma 1. *Let b be a lower bound and B its recursive-extension. For each $(c,E) \in \mathcal{C}^N$, $B(c,E)$ satisfies the following two conditions:*[16]

$$B(c,E) = b(c,E) + B\left(p^b(c,E)\right),$$
$$b(c,E) = 0 \;\Rightarrow\; B(c,E) = 0.$$

Moreover, if b is continuous, the recursive-extension of b is the unique lower bound satisfying the conditions above.

We call the first condition the **recursive condition**, and the second the **no start condition**.

Proof. The fact that the recursive-extension satisfies the conditions is straightforward. Conversely, we show that if b is continuous, it is the unique lower bound satisfying them. Let B and B' satisfy both conditions. Let $(c,E) \in \mathcal{C}^N$. Applying the *recursive* condition to (c,E), we get $B(c,E) - B'(c,E) = B(p^b(c,E)) - B'(p^b(c,E))$. Recursively applying the argument, we construct a bounded and decreasing sequence of problems $(c^k, E^k) = p^b(c^{k-1}, E^{k-1})$. This sequence has a limit $(\underline{c}, \underline{E})$ and, by continuity of B and B',[17] $B(c,E) - B'(c,E) = B(\underline{c}, \underline{E}) - B'(\underline{c}, \underline{E})$. We call the last equality condition $(*)$. Since the sequence is convergent, $b(c^{k-1}, E^{k-1}) \to 0$ and, by continuity of b, $b(\underline{c}, \underline{E}) = 0$. Hence, by *no start*, $B(\underline{c}, \underline{E}) = 0 = B'(\underline{c}, \underline{E})$. Substituting in condition $(*)$, we get $B(c,E) = B'(c,E)$. Since this was done for arbitrary $(c,E) \in \mathcal{C}^N$, $B \equiv B'$. Thus, there is a unique rule satisfying both conditions. □

Even though the recursive-extension of a lower bound is unique, there may be several lower bounds which share the same recursive-extension. In particular, as Corollary 1 shows, if a lower bound is continuous, then its recursive-extension is also its own recursive-extension.

Lemma 2. *Let b be a continuous lower bound and B its recursive-extension. For each $(c,E) \in \mathcal{C}^N$, $B(p^B(c,E)) = 0$ and $b(p^B(c,E)) = 0$.*

Proof. Let $(c,E) \in \mathcal{C}^N$. By continuity, $b(p^B(c,E)) = \lim_{k\to\infty} b(p^{b^k}(c,E))$. By construction of the recursive-extensions, $\{b^k(c,E)\}_{k\in\mathbb{N}}$, $\lim_{k\to\infty} b(p^{b^k}(c,E) = 0$. Hence $b(p^B(c,E)) = 0$ and, by *no start*, $B(p^B(c,E)) = 0$. □

Corollary 1. *Let b be a continuous lower bound and B its recursive-extension. Then, B is its own recursive-extension.*

Proof. Let $(c,E) \in \mathcal{C}^N$. By Lemma 2, $B(p^B(c,E)) = 0$. Hence, for each $(c,E) \in \mathcal{C}^N$, and each $k \in \mathbb{N}$, $B^k(c,E) = B(c,E)$, where B^k is the k-extension of B. Hence, B is its own recursive-extension. □

To illustrate the recursive-extension of a bound, we construct the extensions of *minimal rights* and the *min lower bound*. For the *reasonable lower bound*, we refer to Dominguez and Thomson (2006) for a deeper treatment.

[16] We abuse notation and write $x = 0$ for $(x_1, ..., x_n) = (0, ..., 0)$.
[17] Recall that if a lower bound b is continuous then its recursive-extension B is also continuous.

Proposition 1. *The recursive-extension of minimal rights is minimal rights itself.*

We use the fact that, for each problem, after assigning *minimal rights* and revising the problem accordingly, *minimal rights* of the revised problem are equal to 0 (Thomson 2005a), that is, for each each $(c, E) \in \mathcal{C}^N$, $m(p^m(c, E)) = 0$.

Proof. Let M be the recursive-extension of *minimal rights*. Let $(c, E) \in \mathcal{C}^N$. By the *recursive* condition, $M(c, E) = m(c, E) + M(p^m(c, E))$. Since $m(p^m(c, E)) = 0$, by *no start*, $M(p^m(c, E)) = 0$. Hence, $M(c, E) = m(c, E)$. □

The proof of Proposition 1 illustrates the use of *no start*. Without it, we cannot guarantee that the recursive-extension of the revised problem is 0. Consider the following (discontinuous) bound:[18] let $N = \{1, 2\}$, $\bar{c} = (2, 3)$, $\bar{E} = 1$, and let $A = \{(c, E) \in \mathcal{C}^N \mid p^m(c, E) = (\bar{c}, \bar{E})\}$. Define the bound M' as follows:

$$\begin{aligned} &\text{for each } (c, E) \notin A, \quad M'(c, E) = m(c, E), \\ &\text{for each } (c, E) \in A, \quad M'(c, E) = m(c, E) + (.5, .5). \end{aligned}$$

M' satisfies the *recursive* condition with respect to *minimal rights*, but fails *no start*.

The recursive-extension of the *min lower bound* does not coincide with itself. It assigns to each agent, the minimum of the smallest claim and $\frac{1}{n}$th of the endowment. Several rules satisfy this extension, but for two-agent problems, they coincide with the *constrained equal awards rule* introduced in Section 3.

Proposition 2. *The recursive-extension of the min lower bound assigns, to each agent, the common amount $\lambda \in \mathbb{R}_+$, such that $\lambda = \min\{\{c_i\}_{i \in N}, \frac{E}{n}\}$.*

Proof. The fact that all agents are assigned a common amount λ is a straightforward implication of the fact that, for each problem, the *min lower bound* assigns equal rights to all agents. To prove that $\lambda = \min\{\{c_i\}_{i \in N}, \frac{E}{n}\}$, note that, by *claims boundedness* and *feasibility*, $\lambda \leq \min\{\{c_i\}_{i \in N}, \frac{E}{n}\}$.

Let ν be the recursive-extension of the *min lower bound*. Let $(c, E) \in \mathcal{C}^N$. Since each agent is assigned λ, $(p^\nu(c, E)) = ((c_i - \lambda)_{i \in N}, E - n\lambda)$. If $\lambda < \min\{\{c_i\}_{i \in N}, \frac{E}{n}\}$, then $\frac{1}{n}\min\{\{c'_i\}_{i \in N}, E'\} > 0$, where $(c', E') = p^\nu(c, E)$. Hence, $\mu(p^\nu(c, E)) > 0$, contradicting the conclusion of Lemma 2. □

To see that, for two-agent populations, all rules satisfying the recursive-extension of the *min lower bound* coincide with the *constrained equal awards rule*, consider a problem $(c, E) \in \mathcal{C}^{\{1,2\}}$. Suppose, without loss of generality, that $c_1 < c_2$. If $E \leq 2c_1$, the rights vector is *efficient*, and any rule satisfying the bound divides the endowment equally. If $E > 2c_1$, the bound assigns to each agent c_1, thus any rule satisfying the bound fully honors agent 1's claim and, by *efficiency*, agent 2 gets the remainder. Thus, for each two-agent problem, the awards vector assigned by a rule satisfying the recursive-extension of the *min lower bound* is equal to the awards vector assigned by the *constrained equal awards rule* (see Figure 4).

One may ask when does extending a lower bound single out a rule. The complete answer is given by Theorem 1. For now, we note that *minimal rights* fails *positivity* and so does the *min lower bound*, although, in two-agent problems, the latter satisfies it.

[18] A continuous example can be constructed, but the definition of the lower bound is more involved.

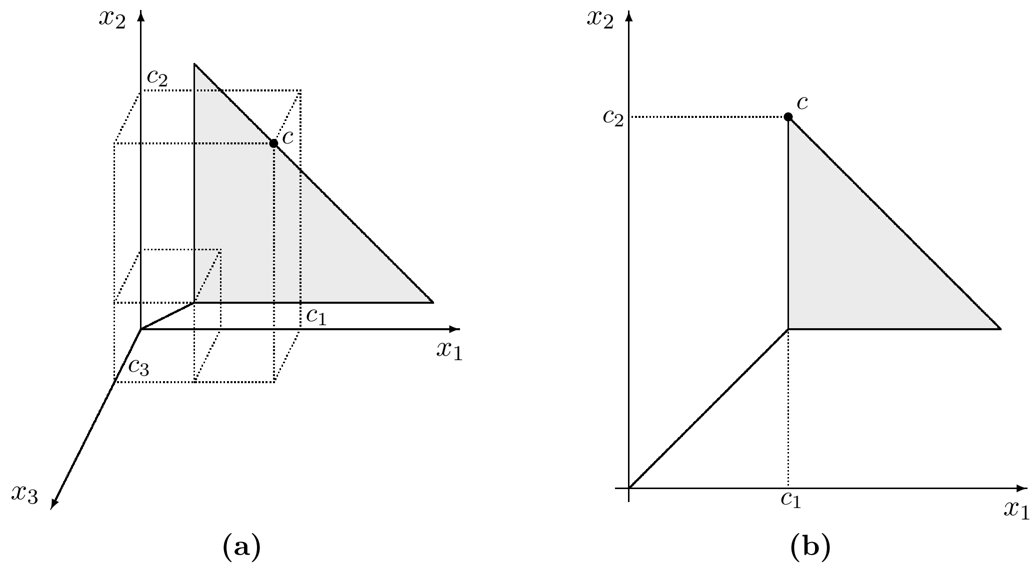

Figure 4: **Graphical representation of the recursive-extension of the *min lower bound*.** (a) Path of acceptable vectors. Many rules satisfy the recursive-extension of the *min lower bound*. (b) Path of acceptable vectors. When $N = \{1, 2\}$, all rules satisfying the bound coincide with the *constrained equal awards rule*.

4.2 An invariance requirement

Given a lower bound consider the requirement that a rule should be obtainable in two ways:

(i) applying the rule directly or,

(ii) first assigning the rights vector, and then applying the rule to the partially resolved problem.

This property was introduced to the literature formulated to *minimal rights* (Curiel, Maschler, and Tijs 1987). For two-agent problems, along with *equal treatment of equals* and *claims truncation invariance*, this property formulated for *minimal rights* characterizes a rule known as the *contested garment rule* (Dagan 1996). We apply the idea to any lower bound:

Definition 5. Given a lower bound b, a rule φ satisfies **invariance under the assignment of $\boldsymbol{b}$**, if for each $(c, E) \in \mathcal{C}^N$,

$$\varphi(c, E) = b(c, E) + \varphi(p^b(c, E)).$$

Many rules satisfy *invariance under the attribution of minimal rights*, but only one satisfies *invariance under the attribution of the reasonable lower bound* (Dominguez and Thomson 2006). One may ask when this invariance property yields a unique rule. Theorem 1 provides the complete answer. For now, we note that the former bound fails *positivity* while the latter satisfies it.

To prove the main theorem, we need the following lemmata:

Lemma 3. *Let B be the recursive-extension of b. For each $(c,E) \in \mathcal{C}^N$, $p^B(c,E) = p^B(p^b(c,E))$.*

Proof. Let $(c,E) \in \mathcal{C}^N$. Recall that, by the *recursive* condition, the recursive-extension satisfies $B(c,E) = b(c,E) + B(p^b(c,E))$.

Then,

$$\begin{aligned} p^B(c,E) &= \left(c - B(c,E), E - \sum_{i\in N} B_i(c,E)\right) \\ &= \left(c - b(c,E) - B\left(p^b(c,E)\right), E - \sum_{i\in N} b_i(c,E) - \sum_{i\in N} B_i\left(p^b(c,E)\right)\right) \\ &= \left(p^b(c,E) - B\left(p^b(c,E)\right), (E - \sum_{i\in N} b_i(c,E)) - \sum_{i\in N} B_i\left(p^b(c,E)\right)\right) \\ &= p^B\left(p^b(c,E)\right). \end{aligned}$$

□

The next lemma provides a simple and useful condition characterizing each lower bound for which there is a unique rule satisfying it. If a unique rule satisfies a bound, then starting from any problem and assigning the rights vector assigned by the bound, the *partially resolved problem* is *degenerate.*

Lemma 4. *Let b be a lower bound, the following statements are equivalent*

(i) There is a unique rule satisfying b.

(ii) For each $(c,E) \in \mathcal{C}^N$, $|X(p^b(c,E))| = 1$.

Proof. To prove that (i) implies (ii), assume that there exist $(c^*,E^*) \in \mathcal{C}^N$ such that $|X(p^b(c^*,E^*))| > 1$. We show that at least two rules satisfy b. Let φ satisfy b, $y = \varphi(c^*,E^*)$, and $x = (y - b(c^*,E^*))$. Then, $x \in X(p^b(c^*,E^*))$. Let $x' \in X(p^b(c^*,E^*))$ be such that $x' \neq x$ and $y' = x' + b(c^*,E^*)$. Then, $y' \neq y$ and $y' \in X(c^*,E^*)$. Let $\phi(c^*,E^*) = y'$ and, for each $(c,E) \neq (c^*,E^*)$, let $\phi(c,E) = \varphi(c,E)$. Then, $\phi \neq \varphi$ and both φ and ϕ satisfy b.

Now we prove that, if for each $(c,E) \in \mathcal{C}^N$, $|X(p^b(c,E))| = 1$, then there is a unique rule satisfying b. Let $(c,E) \in \mathcal{C}^N$, φ a rule satisfying b, and $x = \varphi(c,E) - b(c,E)$. By *claims boundedness*, $x \leq c - b(c,E)$. Since φ satisfies b, $x \geq 0$. Summing over agents, $\sum_{i\in N} x_i = \sum_{i\in N} \varphi_i(c,E) - \sum_{i\in N} B_i(c,E) = E - \sum_{i\in N} B_i(c,E)$. These three conditions imply that $x \in X(p^b(c,E))$. Since $|X(p^b(c,E)| = 1$, $x = X(p^b(c,E))$. Thus, $\varphi(c,E) = b(c,E) + X(p^b(c,E))$. □

Corollary 2. *Let b be a lower bound and φ the unique rule satisfying it. For each $(c,E) \in \mathcal{C}^N$,*

$$\varphi(c,E) = b(c,E) + X(p^b(c,E).$$

Proof. Direct implication of the proof of Lemma 4. □

We are now ready for the main theorem:

Theorem 1. *Let b be a continuous lower bound. The following conditions are equivalent:*

(i) b satisfies positivity.

(ii) There is a unique rule satisfying the recursive-extension of b.

(iii) There is a unique rule satisfying invariance under the assignment of b.

Moreover, the rules in (ii) and (iii) coincide.

Proof. First, we show that (i) implies (ii). Let B be the recursive-extension of b. Let $(c, E) \in \mathcal{C}^N$. By Lemma 2, $b(p^B(c, E)) = 0$. Hence, by *positivity*, $p^B(c, E)$ is *degenerate*, and therefore $|X(p^B(c, E))| = 1$. Since the argument holds for arbitrary $(c, E) \in \mathcal{C}^N$, by Lemma 4, there is a unique rule satisfying B.

Next, we prove that (ii) implies (iii), and that the unique rule satisfying the recursive-extension of b coincides with the unique rule satisfying *invariance under the assignment of* b. Let B be the recursive-extension of b and φ the unique rule satisfying B. First, we show that φ satisfies *invariance under the assignment of* b. Then, we show that no other rule does.

Let $(c, E) \in \mathcal{C}^N$. Since $\varphi(c, E)$ is the unique rule satisfying B, by Corollary 2, $\varphi(c, E) = B(c, E) + X(p^B(c, E))$. By the *recursive* condition $\varphi(c, E) = b(c, E) + B(p^b(c, E)) + X(p^B(c, E))$. By Lemma 3, $\varphi(c, E) = b(c, E) + B(p^b(c, E)) + X(p^B(p^b(c, E)))$. By Corollary 2 applied to $p^b(c, E)$, $\varphi(c, E) = b(c, E) + \varphi(p^b(c, E))$. Thus, φ satisfies *invariance under the assignment of* b.

Now we prove that no other rule satisfies *invariance under the assignment of* b. Let B be the recursive-extension of b, φ the unique rule satisfying B, and ϕ a rule satisfying *invariance under the assignment of* b. Let $(c, E) \in \mathcal{C}^N$. Since ϕ satisfies *invariance under the assignment of* b, $\phi(c, E) = b(c, E) + \phi(p^b(c, E)$. Repeated application of *invariance under the assignment of* b yields $\phi(c, E) = B(c, E) + \phi(p^B(c, E))$. By Lemma 4, $|X(p^B(c, E))| = 1$. Thus, $\phi(c, E) = B(c, E) + X(p^B(c, E))$. By Corollary 2, $\phi(c, E) = \varphi(c, E)$. Since the argument holds for arbitrary $(c, E) \in \mathcal{C}^N$, $\phi \equiv \varphi$.

Finally, we show that (iii) implies (i). Let φ be the unique rule satisfying *invariance under the assignment of* b. Let $(\bar{c}, \bar{E}) \in \mathcal{C}^N$ be *non-degenerate*. We show that $b(\bar{c}, \bar{E}) \geq 0$.

Suppose, by contradiction, that $b(\bar{c}, \bar{E}) = 0$. Let $x \in X(\bar{c}, \bar{E})$ be such that $x \neq \varphi(\bar{c}, \bar{E})$. Since $(\bar{c}, \bar{E})$ is *non-degenerate*, such x exists. Let $A^0 = \{(\bar{c}, \bar{E})\}$, and for each $k \geq 1$, let $A^k = \{(c, E) \mid p^b(c, E) \in A^{k-1}\}$. It is easy to show that $A^{k-1} \subseteq A^k$. Let $A \subseteq \mathcal{C}^N$ be the smallest set such that, for each $k \in \mathbb{N}$, $A^k \subseteq A$. For each $(c, E) \in \mathcal{C}^N$ define the rule ϕ by:

$$(c, E) \in A \Rightarrow \phi(c, E) = b(c, E) + x,$$

$$(c, E) \in A^c \Rightarrow \phi(c, E) = \varphi(c, E).$$

Then, $\phi \neq \varphi$ and ϕ satisfies *invariance under the attribution of* b, a contradiction with uniqueness of such a rule. Thus, $b(\bar{c}, \bar{E}) \geq 0$. □

It is worth noting that, if a lower bound fails *positivity*, there are rules satisfying its recursive-extension, but failing *invariance under the assignment* of the bound. For example, consider *minimal rights*; as we have seen, *minimal rights* is its own recursive-extension (Proposition 1). Moreover, all rules satisfy *minimal rights*. Hence, all rules satisfy the recursive-extension of *minimal rights*. On the other hand, many rules fail *invariance under* *the assignment of minimal rights*. An example is the *proportional* rule. This example shows that, if we drop *positivity*, satisfying the recursive-extension of a bound and satisfying *invariance under the assignment of the bound* are not equivalent.

A straightforward corollary of Theorem 1 is that there is a unique rule satisfying *invariance under the assignment of the reasonable lower bound* (Dominguez and Thomson 2006). By the following corollary, we can also conclude that the recursive-extension of the *reasonable lower bound* is such a rule.

Corollary 3. *If a continuous lower bound b satisfies positivity and conditional efficiency, then its recursive-extension is efficient. Thus, the recursive-extension of b defines the unique rule satisfying invariance under the assignment of b.*

Proof. Let b satisfy *positivity* and *conditional efficiency*, B its recursive-extension, and φ a rule satisfying B. Let $(c, E) \in \mathcal{C}^N$. We show that $B(c, E) = \varphi(c, E)$. By Lemma 2, $b(p^B(c, E)) = 0$. Hence, by *positivity*, $|X(p^B(c, E))| = 1$. By *conditional efficiency*, $b(p^B(c, E)) = X(p^B(c, E))$. Thus, $B(p^B(c, E)) = X(p^B(c, E))$. By Lemma 2, $B(p^B(c, E)) = 0$. Thus, $X(p^B(c, E)) = 0$. By Theorem 1, $\varphi(c, E) = B(c, E) + X(p^B(c, E))$. Thus, $B(c, E) = \varphi(c, E)$. □

This result can also be interpreted in the following way: suppose that a lower bound satisfies *conditional efficiency*. Then, its recursive-extension "inherits" *conditional efficiency*. Moreover, if the bound also satisfies *positivity*, its recursive-extension "inherits" *efficiency*. In the next section we ask this sort of questions for some properties.

A converse statement of Corollary 3 holds by weakening *conditional efficiency* in the following way: Let $\varepsilon > 0$. A lower bound b satisfies **ε-conditional efficiency** if, for each *degenerate problem* $(c, E) \in \mathcal{C}^N$, $\sum_{i \in N} b_i(c, E) \geq \varepsilon E$. Clearly, if $\varepsilon > 1$ the condition cannot be satisfied. If $\varepsilon = 1$ the condition corresponds to *conditional efficiency*. If $\varepsilon < 1$ the condition is weaker than *conditional efficiency*. It states that, for a *degenerate problem* with a positive endowment, the agent with a positive claim should receive a positive right.

Corollary 4. *Let b be a continuous lower bound and $\varepsilon > 0$. If b satisfies ε-conditional efficiency and positivity, then its recursive-extension is efficient. Conversely, if the recursive-extension of b is efficient, then for some $\varepsilon > 0$, b satisfies ε-conditional efficiency and positivity.*

Proof. The proof of the first statement follows the same logic as Corollary 3. We prove the converse statement. Let B be the recursive-extension of b. If B is *efficient*, there is a unique rule satisfying B. By Theorem 1, b satisfies *positivity*. Assume, by contradiction, that for each $\varepsilon > 0$, b fails *ε-conditional efficiency*. Then, there exists a *degenerate problem* $(c, E) \in \mathcal{C}^N$ such that $b(c, E) = 0$ and $X(c, E) \geq 0$.[19] By *no start*, $B(c, E) = 0$. Hence, B is not *efficient*, contradicting our initial hypothesis. Thus, there is $\varepsilon > 0$ such that b satisfies *ε-conditional efficiency*. □

5 Inheritance of properties

If a lower bound is well-behaved, is its recursive-extension also well-behaved? Now, we introduce some properties of good behavior of lower bounds, their desirability can be evaluated

[19]The existence of a *degenerate problem* (c, E) with $b(c, E) = 0$ is implied by the closedness of the set of *degenerate problems* and the continuity of b.

from different points of view: horizontal equity, monotonicity, and invariance properties. These properties are used as tests of good behavior for the lower bounds. Many desirable properties for rules are also meaningful and desirable for lower bounds. Conversely, many desirable properties for lower bounds are desirable for rules. For instance, in most applications, it is desirable that agents with equal claims be treated equally; *equal treatment of equals* says that a rule assigns equal awards to agents with equal claims; in such situations, it also seems reasonable that a lower bound assigns equal rights to those agents. We introduce some of the properties that have been discussed in the literature for rules, and note which of the lower bounds introduced in Section 3 satisfy them. Straightforward proofs are omitted.

Then, we ask whether the recursive-extension of a lower bound satisfies the same properties as the original bound. For some properties this is the case, and we say that the properties are **inherited by the recursive-extension**. Some properties are inherited on their own (directly). Other properties, which are not directly inherited, are inherited if other properties are imposed as well. We call such cases **assisted inheritance**,[20] since the additional properties imposed assist the original property to be inherited. We undertake a systematic investigation of inheritance of some properties. Some proofs are straightforward and we do not provide them. We relegate most examples for negative results to the appendix. It is worth noting that the properties are inherited by the recursive-extension, and not necessarily by rules satisfying the recursive-extension. When a lower bound satisfies positivity, the unique rule satisfying the recursive-extension usually inherits properties.[21]

Definition 6. A property is **inherited** (by the recursive-extension of a bound) if, whenever a bound satisfies the property, its recursive-extension also does.

We start with properties that can be interpreted as expressing the idea of horizontal equity. First, agents with equal claims should be assigned equal rights:

Equal treatment of equals: For each $(c, E) \in \mathcal{C}^N$, and each $\{i, j\} \in N$, if $c_i = c_j$, $b_i(c, E) = b_j(c, E)$.

Minimal rights, the *reasonable lower bound*, and the *min lower bound* satisfy *equal treatment of equals.*

For each problem, a bound which satisfies *equal treatment of equals* assigns equal rights to agents with equal claims. In the revised problem, their claims are equal and they are assigned equal rights. Thus, the revised bound satisfies *equal treatment of equals.* A recursive argument shows that *equal treatments of equals* is inherited.

The same logic can be used to prove inheritance of most properties: we start with a problem that satisfies the hypotheses of the property, assign the rights vector and revise the problem accordingly; we check whether the conclusions of the property imply that the revised problem satisfies its hypotheses. If it does, we add the rights vector of the revised problem to the rights vector of the original problem, and check if the revised bound satisfies

[20] We borrow the term "assisted" from Hokari and Thomson (2006). They use the expression in their study of *consistency* and its implications. A property is *lifted* by consistency if, when the property is satisfied by a rule for two-agent populations, it is also satisfied for all populations provided the rule is consistent. They call *assisted lifting* if a property is not lifted by consistency on its own, but it is lifted provided that the rule satisfies some additional properties.

[21] Recall that by Corollary 4, if for some $\varepsilon > 0$ a lower bound satisfies *positivity* and *ε-conditional efficiency*, then its recursive-extension defines a rule and it inherits the properties. For some properties, even if a *positive* lower bound fails *ε-conditional efficiency*, the unique rule satisfying its recursive-extension inherits the properties.

the conclusions of the property. If it does, a recursive argument proves inheritance of the property.

The next property implies *equal treatment of equals*. The lower bound should not depend on the names of the agents. The lower bounds introduced in Section 3 satisfy this property. Denote by Π^N the set of permutations on N:

Anonymity: For each $(c, E) \in \mathcal{C}^N$, and each permutation $\pi \in \Pi^N$, $b(\pi(c), E) = \pi(b(c, E))$.

It is easy to see that *anonymity* is inherited. But not all properties are inherited and careful examination is required. The next property also implies *equal treatment of equals* but it is logically independent of *anonymity*. If agent i's claim is at least as large as agent j's, agent i's right should be at least as large as agent j's (Aumann and Maschler 1985).

Order preservation: For each $(c, E) \in \mathcal{C}^N$, and each pair $\{i, j\} \subseteq N$, if $c_i \geq c_j$, $b_i(c, E) \geq b_j(c, E)$.

In fact, as the following example shows, not even the basic property of *order preservation* is inherited:

Let $N = \{1, 2\}$. Consider the bound b defined as follows: for the problem $(\bar{c}_1, \bar{c}_2, \bar{E}) = (2, 2.5, 2.5)$, $b(2, 2.5, 2.5) = (0, 1)$. For the problem $(\bar{c}'_1, \bar{c}'_2, \bar{E}') = (2, 1.5, 1.5)$, $b(2, 1.5, 1.5) = (1.5, 0)$. For each other problem $(c, E) \in \mathcal{C}^N$, $b(c, E) = 0$. The recursive-extension of the bound is: $B(2, 2.5, 2.5) = (1.5, 1)$, $B(2, 1.5, 1.5) = (1.5, 0)$, and for each other problem $(c, E) \in \mathcal{C}^N$, $B(c, E) = 0$. Then, b satisfies *order preservation* and B fails it. Hence, the property is not inherited.

In the literature, *order preservation* (of awards) is usually imposed together with a *dual* property on the losses experienced by the agents.[22] Given a problem and an awards vector, each agent's loss is equal to the difference between her claim and her award. In our setting, a lower bound (on awards) has a *dual lower bound on losses*. Given a lower bound b, for each $(c, E) \in \mathcal{C}^N$, $b^d(c, E) = c - b(c, E)$ defines such a bound. Agents with smaller claims should be assigned smaller lower bounds on losses:[23]

Order preservation of losses: For each $(c, E) \in \mathcal{C}^N$, and each pair $\{i, j\} \subseteq N$, if $c_i \geq c_j$, then, $b^d_i(c, E) \geq b^d_j(c, E)$.

An example dual to the one above shows that *order preservation of losses* is not inherited. When a lower bound satisfies both *order preservation* and *order preservation of losses* we say that it satisfies **full order preservation**. The three bounds presented in Section 3 satisfy *full order preservation*.

Given that the above example is not an intuitive lower bound and, in particular, fails *order preservation of losses*, one may wonder if there are well-behaved bounds for which the property is not inherited. The next proposition shows that this is not the case.

Proposition 3. *Full order preservation is inherited.*

Proof. Let b satisfy *full order preservation*. For each $(c, E) \in \mathcal{C}^N$ let $b'(c, E) = b(c, E) + b(p^b(c, E))$. We show that b' satisfies *full order preservation*. Let $(c, E) \in \mathcal{C}^N$, and $\{i, j\} \subseteq N$. Without loss of generality assume $c_i < c_j$. By *full order preservation*, $b_i(c, E) < b_j(c, E)$ and $c_i - b_i(c, E) < c_j - b_j(c, E)$. Consider the problem $(c', E') = p^b(c, E)$. By *order preservation of losses*, $c'_i < c'_j$. By *full order preservation*, $b_i(c', E') < b_j(c', E')$ and $c'_i - b_i(c', E') < c'_j - b_j(c', E')$. Then, $b'_i(c, E) < b'_j(c, E)$ and $c_i - b'_i(c, E) < c_j - b'_j(c, E)$. Hence,

[22] For a systematic treatment of duality see Thomson and Yeh (2005).

[23] A smaller lower bound on her losses is better for an agent. *Order preservation of losses* is a way to prevent agents with smaller claims form being treated too harshly.

b' satisfies *full order preservation*. A recursive argument shows that *full order preservation* is inherited. □

We turn to monotonicity properties. They are relational conditions. If an agent's claim increases, her right should not decrease:

Claims monotonicity: For each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, if $c_i < c'_i$, $b_i(c, E) \leq b_i(c'_i, c_{-i}, E)$.[24]

We may also be interested in how an increase in an agent's claim affects the rights of the others. When an agent's claim increases, it is natural to require that the other agent's rights should not increase:

Others-oriented claims monotonicity: For each $(c, E) \in \mathcal{C}^N$ and each $i \in N$, if $c_i < c'_i$, $b_{N\setminus\{i\}}(c, E) \geq b_{N\setminus\{i\}}(c'_i, c_{-i}, E)$.

When the endowment increases, agents should not be assigned smaller rights:

Resource monotonicity: For each $(c, E) \in \mathcal{C}^N$, each $i \in N$, and each $E' < E$, $b(c, E') \leq b(c, E)$.

The three bounds presented in Section 3 satisfy *claims monotonicity* and *resource monotonicity*. The *min lower bound* does not satisfy *others-oriented claims monotonicity*, but the other two bounds do.

Most monotonicity properties are not directly inherited. In the Appendix, we provide examples of bounds to show these negative results. The intuition for the failures is the following: start with a pair of problems (c, E) and $(\bar{c}, \bar{E})$ related as in the hypotheses of the property; assign the rights vector and revise the problems accordingly; since, for each problem, the claims and the endowment are revised, the resulting pair of problems may fail the monotonicity hypotheses. In this case, the lower bound can assign a rights vector that fails the monotonicity conclusions for the original problem, and when summing the rights vectors of the two problems, the resulting revised bound can fail the monotonicity conclusions.

An interesting question is whether or not monotonicity properties are inherited when several of them are imposed at the same time, by bounding the gains and losses from changes in the data,[25] or when dual monotonicity properties on the losses are imposed. As we saw for *order preservation*, its dual property assisted its inheritance. Moreover, as with most dual properties, dual monotonicity properties have intuitive appeal. We conjecture that, if "enough monotonicity" is imposed, a positive answer can be obtained. In fact, for the *reasonable lower bound*, positive answers for inheritance of several monotonicity properties exist (Dominguez and Thomson 2006).

We turn to invariance properties. Since an agent cannot get more than the endowment, the rights vector should not be affected if we truncate the claims at the endowment. All three bounds in Section 3 satisfy the property:

Claims truncation invariance: For each $(c, E) \in \mathcal{C}^N$ $b(c, E) = b(t(c, E), E)$.

For *invariance under claims truncation* we have a positive inheritance result:

Proposition 4. *Invariance under claims truncation is inherited.*

[24] Let $(c, E) \in \mathcal{C}^N$, and $N' \subseteq N$. The vector $c_{N'}$ is the vector c restricted to the population N'. The vector c_{-i} is the vector c restricted to the population $N \setminus \{i\}$. The vector $(c_{N\setminus N'}, c'_{N'})$ is the vector c where the N'th coordinates have been replaced by $c'_{N'}$.

[25] Bounds on the gains and losses from changes in the data are intuitive. For instance, when an agents' claim increases, we could require that her right does not increase by more than the increase in her claim. Some bounds on gains and losses can be found in Thomson (2005a).

Proof. Let b satisfy *invariance under claims truncation.* Let $(c, E) \in \mathcal{C}^N$. Let $\bar{c} = t(c, E)$ denote the vector of truncated claims and $(\bar{c}, \bar{E}) = (t(c, E), E)$ the truncated problem. By *invariance under claims truncation,*

$$b(c, E) = b(\bar{c}, \bar{E}). \tag{1}$$

Let $(c', E') = p^b(c, E)$ and $(\bar{c}', \bar{E}') = p^b(t(c, E), E)$ denote the *b-partially resolved problems.* We claim that $(t(c', E'), E') = \big(t(\bar{c}', \bar{E}'), \bar{E}'\big)$. Hence, by *invariance under claims truncation* applied to both problems,

$$b(c', E') = b(\bar{c}', \bar{E}'). \tag{2}$$

To prove the claim, we first note that, by definition, $E = \bar{E}$. Hence, by (1), $E' = \bar{E}'$. Now we show that for each $i \in N$, $t_i(c', E') = t_i(\bar{c}', \bar{E}')$. Let $i \in N$. We distinguish two cases:

(i) If $c_i \leq E$, then $\bar{c}_i = c_i$ and $c_i - b_i(c, E) = \bar{c}_i - b_i(c, E)$. Using (1), $c_i - b_i(c, E) = \bar{c}_i - b_i(\bar{c}, \bar{E})$. Hence, $c'_i = \bar{c}'_i$, and since $E' = \bar{E}'$, by definition of the truncation operator, $t_i(c', E') = t_i(\bar{c}', \bar{E}')$.

(ii) If $E < c_i$, then $E - \sum_{j \in N} b_j(c, E) < c_i - b_i(c, E)$. Hence, $E' < c'_i$ and $t_i(c', E') = E'$. We call the last equality condition $(*)$. Moreover, $E < c_i$ implies $\bar{c}_i = E$. Since $E = \bar{E}$, we have $\bar{c}_i - b_i(\bar{c}, \bar{E}) = \bar{E} - b_i(\bar{c}, \bar{E}) \geq \bar{E} - \sum_{j \in N} b_j(\bar{c}, \bar{E})$. Hence, $\bar{c}'_i \geq \bar{E}'$ and $t_i(\bar{c}', \bar{E}') = \bar{E}'$. By condition $(*)$ and since $E' = \bar{E}'$, $t_i(\bar{c}', \bar{E}') = t_i(c', E')$.

This completes the proof of the claim.

Now, summing equations (1) and (2),

$$b'(c, E) = b(c, E) + b(c', E') = b(\bar{c}, \bar{E}) + b(\bar{c}', \bar{E}') = b'(t(c, E), E).$$

Hence b' satisfies *invariance under claims truncation.* A recursive argument shows that *invariance under claims truncation* is inherited. □

A direct corollary of Proposition 4 is that the unique rule satisfying the *reasonable lower bound* satisfies *invariance under claims truncation* (Dominguez and Thomson 2006).

Appendix

Examples of inheritance failure for monotonicity properties.

We provide a two-agent example for *claims monotonicity.* The bound satisfies the property, but its *recursive-extension* fails it. For two-agent problems *claims monotonicity* and *others-oriented claims monotonicity* are equivalent. Hence, this example also shows inheritance failure of *others-oriented claims monotonicity.* We provide the rights vectors assigned by the bound and by its recursive-extension for selected problems. The bound should be defined for all problems in a way that satisfies the property of interest. In the tables, the problems (c, E) and $(\bar{c}, \bar{E})$ are the initial problems, and they satisfy the monotonicity hypotheses. The problems (c', E') and $(\bar{c}', \bar{E}')$ are the revised problems $p^b(c, E)$ and $p^b(\bar{c}, \bar{E})$ respectively.

1. ***Claims monotonicity* and *others oriented claims monotonicity*:**

(c_1, c_2, E)	$(30, 90, 100)$	$(\bar{c}_1, \bar{c}_2, \bar{E})$	$(30, 100, 100)$
$b(c_1, c_2, E)$	$(10, 30)$	$b(\bar{c}_1, \bar{c}_2, \bar{E})$	$(10, 45)$
(c_1', c_2', E')	$(20, 60, 60)$	$(\bar{c}_1', \bar{c}_2', \bar{E}')$	$(20, 55, 45)$
$b(c_1', c_2', E')$	$(10, 50)$	$b(\bar{c}_1', \bar{c}_2', \bar{E}')$	$(20, 25)$
$B(c_1, c_2, E)$	$(20, 80)$	$B(\bar{c}_1, \bar{c}_2, \bar{E})$	$(30, 70)$
$B(c_1', c_2', E')$	$(10, 50)$	$B(\bar{c}_1', \bar{c}_2', \bar{E}')$	$(20, 25)$

2. ***Resource monotonicity*:**

(c_1, c_2, E)	$(30, 90, 100)$	$(\bar{c}_1, \bar{c}_2, \bar{E})$	$(30, 90, 110)$
$b(c_1, c_2, E)$	$(10, 30)$	$b(\bar{c}_1, \bar{c}_2, \bar{E})$	$(15, 75)$
(c_1', c_2', E')	$(20, 60, 60)$	$(\bar{c}_1', \bar{c}_2', \bar{E}')$	$(15, 15, 20)$
$b(c_1', c_2', E')$	$(20, 40)$	$b(\bar{c}_1', \bar{c}_2', \bar{E}')$	$(10, 10)$
$B(c_1, c_2, E)$	$(30, 70)$	$B(\bar{c}_1, \bar{c}_2, \bar{E})$	$(25, 85)$
$B(c_1', c_2', E')$	$(20, 40)$	$B(\bar{c}_1', \bar{c}_2', \bar{E}')$	$(10, 10)$

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