Measuring gains from trade and an application to fair allocation

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Abstract

In the classical model of exchange, gains from trade can be obtained if there exists a feasible allocation which each agent prefers to her endowment. There is no single accepted way of measuring these gains in terms of welfare, for interpersonal comparisons of welfare are needed. In this paper we propose a method of measuring gains from trade in terms of quantities of goods, avoiding welfare comparisons. The measure is given by the amount of resources that can be saved, relative to the aggregate endowment, while keeping each agent's welfare unchanged. Then, based on Shapley's algorithm, we propose a way of distributing gains from trade fairly among the agents. Since fair distribution of gains from trade can be inefficient, we show that a recursive procedure, which is fair at each step of the recursion, yields an efficient allocation.

Keywords: Fair allocation, Gains from trade, Recursive methods.
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1 Introduction

In the classical model of exchange gains from trade can be obtained if there exists a feasible allocation which each agent prefers to her endowment. But, how can we measure the gains from trade in an economy? For interpersonal comparisons of welfare are typically not meaningful, we propose measuring gains from trade in terms of

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quantities of goods, avoiding welfare comparisons. To do so, we search for a “reference allocation”, composed of “reference bundles”, one for each agent, such that: (i) each agent is indifferent between her endowment and her reference bundle, and (ii) the reference allocation is feasible. Since no welfare gains are achieved, the difference between the aggregate endowment and the resources at the reference allocation provides a measure of the gains from trading from the endowment profile to the reference allocation.

In this manner, we obtain a measure of gains from trade in terms of quantities of goods. But, for most economies, there is a continuum of reference allocations and the vectors of resources saved by trading to each differ. The set of all such vectors defines the set of possible gains from trade of the economy. We introduce the notion of a (vector-valued) “metric”, to select, for each economy, one representative vector from its set of possible gains from trade.

Two economies may differ in preferences and endowment profiles but have equal sets of possible gains from trade. Our premise is that, if two economies have equal sets of possible gains from trade, a metric should not distinguish between them, and should select the same representative vector of gains from trade in both economies. Thus, the notion of a metric is similar to the notion of a solution for bargaining problems (Nash 1950). A “bargaining problem” consists of a set of utility profiles and a disagreement point; a “solution” maps each bargaining problem into a utility profile. In our setting, the set of utility profiles corresponds to the set of possible gains from trade, and the disagreement point corresponds to each agent consuming her endowment and no resources being saved. A metric maps each set of possible gains from trade into a vector of gains from trade.

We follow an approach used in bargaining theory and look for metrics satisfying certain desirable properties.[1] Metrics should select a vector representative of the size of the set of possible gains from trade. Thus, we look for metrics satisfying the following three properties that have strong intuitive appeal for our setting. The first property is “maximality”: for each economy, no reference allocation leads to a larger vector of gains from trade than the vector selected by the metric. The second property is “monotonicity” with respect to set inclusion: a metric selects a larger vector of gains from trade in an economy with a larger set of possible gains from trade. The third property is “homogeneity”: a homogeneous expansion of the set of possible gains from trade leads to a homogeneous expansion of the vector selected by

[1] For a discussion on the axiomatic method applied to economic problems see Thomson (2001).

the metric.[2]

We show that if a metric satisfies maximality, monotonicity, and homogeneity, then it is a member of a "weighted-gains family" of metrics. Each member of this family is described by a vector of weights, one for each good. Weights represent the relative importance of each good in measuring gains from trade. For each economy and each vector of weights α, the α-weighted-gains metric measures gains from trade by the largest vector of possible gains from trade proportional to α.

The choice of the vector of weights can be model-specific. For example, for economies where each agent's preferences are quasi-linear with respect to one good (e.g. contract theory), the natural choice is to assign full weight to the numeraire good. For economies where some goods are deemed more important than others (e.g. necessities vs. luxuries) placing relatively larger weights on these goods may be desirable. For a multi-period economy where agents consume positive amounts of a single good per period, measuring gains from a smooth consumption stream in terms of consumption during the initial period seems a reasonable choice. For economies where agents face idiosyncratic uncertainty but there is no aggregate uncertainty, measuring gains from perfect insurance leads to a choice of equal weights. For the full domain of economies, symmetry with respect to goods leads to a choice of equal weights.

Once we have a measure of the gains from trade in an economy, how can we distribute them fairly? We propose to use the solution concept of the theory of cooperative games known as the Shapley-value, using its interpretation as rewarding agents as a function of their "marginal contributions" to all subgroups.[3] Imagine agents arriving one at a time and calculate, for each agent, the difference between: (i) the gains in the economy consisting of her and all agents preceding her, and (ii) the gains in the economy consisting of all agents preceding her. We take the average of the differences over all possible orders of arrival, under the premise that all orders are equally likely, as a measure of her contribution to the gains from trade.

Finally, we apply our definitions to obtain a fair allocation. An allocation is fair if each agent receives her contribution to the gains from trade. First, we consider assigning to each agent the bundle obtained as the sum of: (i) her bundle in the reference allocation selected to measure gains from trade, and (ii) her contribution

[2]Other properties from the theory of bargaining also have appeal in our setting. So do other solutions that can be interpreted as metrics. For the application to fair allocation, the proofs rely on monotonicity of the metric, but alternative proofs based on weaker requirements can be obtained. In Section 5 we discuss relaxing monotonicity and how our results change.

[3]We do not provide a characterization for this method of measuring contributions. An open question for future research is to provide properties for pairs of metrics and measures of contributions, and study their implications.

to the gains from trade. The allocation so obtained is fair, feasible, and exhausts resources. But it may be inefficient. If it is inefficient, the gains from trade in the resulting economy are positive, and it is natural to distribute them fairly using the same procedure. We show that proceeding recursively yields an efficient allocation.

1.1 Related literature

Our proposal for measuring gains from trade in terms of quantities of goods can be interpreted as a generalization, to a multi-agent setting, of some existing measures of welfare changes in single agent decision making settings. The equivalent variation and compensating variation are measures of welfare changes in terms of the difference in expenditure required to keep an agent's welfare unchanged after a change in prices. In our setting, change does not come from prices but from trading among agents, and measuring gains in terms of quantities of goods seems natural in the absence of pre-specified prices. In settings of choice under uncertainty, the risk premium measures how much an agent is willing to forgo in order to obtain a constant consumption stream; the certainty equivalent measures the level of constant consumption across states that leaves the agent's welfare unchanged. In our setting, we measure how much a set of agents can gain by redistributing risk among them.

Measuring gains from trade is equivalent to measuring the inefficiency of the endowment. A measure the inefficiency of an allocation (or of the endowment profile) is its "coefficient of resource utilization" (Debreu 1951). It assigns to each maximal vector in the set of possible gains from trade a number equal to the dot product of the vector and its supporting price.[4] Then, it measures the inefficiency of the allocation by the maximal such value.

This way of measuring the inefficiency of an allocation is similar to ours. It also considers the set of possible gains from trade of the economy. But, instead of measuring gains from trade by a vector of commodities, it measures gains from trade by a scalar. Using a real-valued metric implies that we can order the set of all economies according to their gains from trade. Using a vector-valued metric allows for a partial order, which may be desirable if differences in goods require asymmetric treatment across them.[5] Moreover, this measure is not monotonic, an increase in the

[4] The set of possible gains from trade is a convex set. Hence, at each maximal element of this set there exist a vector of supporting prices. Moreover, this price vector also supports each agent's preferences at her reference bundle in the reference allocation leading to such gains from trade.

[5] As our results show, if a vector-valued metric satisfies maximality, monotonicity, and homogeneity, then, it implies the existence of an order on the set of economies according to their gains from trade.

set of possible gains from trade can lead to a decrease in the measurement of this gains.

Another advantage of using a vector-valued metric over a real-valued one, is that a vector-valued metric leads to a natural allocation at which gains from trade are distributed fairly. The theory of fair allocation can be categorized according to the nature of the problem under study: First, situations where a social endowment has to be divided among a set of agents. Second, situations where agents have private endowments and redistribution (trading) is possible. For the problem of allocating a social endowment two notions of fairness are prominent. First is no-envy (Foley 1967): no agent should prefer another agent's bundle over her own (see Kolm (1998) and Varian (1976)). Second is egalitarian equivalence (Panzer and Schmeidler 1978): there exists a reference bundle such that each agent is indifferent between her bundle and the reference bundle. For the problem of redistributing individual endowments these two notions can be adapted. No-envy in trades states that no agent prefers another agent's trade over her own. Egalitarian-equivalence from endowments states that there exists a reference vector such that each agent is indifferent between her bundle and the bundle obtained from the sum of her endowment and the reference vector.

Recently, a notion similar to egalitarian equivalence was proposed for economies with individual endowments: an allocation is fair if it is welfare equivalent to an allocation obtained from summing to the endowment profile a vector of fair "concessions" (Pérez-Castrillo and Wettstein 2006). This notion generalizes egalitarian equivalence in two ways: first, it allows for differences in the reference bundles according to differences in individual endowments; second, it allows for differences in concessions.

Our notion of fairness is similar to Pérez-Castrillo and Wettstein (2006) but it differs in two ways. First, our reference allocation is welfare equivalent to the endowment profile, and we sum to the reference allocation the vector of contributions. Second, our vector of contributions differs from their vector of concessions. Also, our results differ in form from theirs. They show existence of fair and efficient allocations; we do not obtain a fair and efficient allocation immediately, but propose a recursive procedure which is fair at each step, and obtains an efficient allocation at the limit. Also, we provide an algorithm to reach it.[6]

The paper is organized as follows: In the next section we present the model. In Section 3 we study how to measure gains from trade. Section 4 contains the

[6]For the class of quasi-linear economies, taking the vector of weights assigning full weight to the numeraire good leads to coincidence between both notions of fairness.

application to fair allocation. Finally, in Section 5 we conclude and discuss some open questions. In particular, we discuss relaxing some restrictions on the domain of preferences and some of the properties of metrics.

2 The Model

There is a set $N = \{1, 2, ..., n\}$ of agents and a set $M = \{1, 2, ..., m\}$ of goods. For each $i \in \mathbb{N}$, i's **consumption set** is $X_i = \mathbb{R}^m_+$.[7] We refer to agent i's consumption vector as her bundle. Each $i \in N$ has a complete, transitive, continuous, strictly monotonic in the interior of the consumption set, and convex **preference relation** R_i over pairs of bundles in X_i.[8] For simplicity, we assume that for each pair $x,\ x' \in X_i$, if $x > 0$ and there exists $k \in M$ such that $x'_k = 0$,[9] then $x\ P_i\ x'$.[10] We call this condition **boundary aversion**. The set of all such preferences is denoted $\mathcal{R}$. A profile of preferences is a list $R = (R_i)_{i\in N} \in \mathcal{R}^n$. Each agent has an **endowment** of goods $\omega_i \in \mathbb{R}^m_{++}$. An endowment profile is a list $\omega = (\omega_i)_{i\in N}$. The set $\mathcal{E} = \mathcal{R}^n \times \mathbb{R}^{mn}_{++}$ is the set of all economies.

An **allocation** $x = (x_i)_{i\in N}$ assigns to each $i \in N$ the bundle $x_i \in X_i$. The set of allocations is denoted $X = \bigtimes_{i\in N} X_i$. Given an allocation $x \in X$, the **resources at $\boldsymbol{x}$** are denoted $\Sigma(x) = \sum_{i\in N} x_i$. Similarly, given a set of allocations $S \subset X$, $\Sigma(S) = \{y \in \mathbb{R}^m_+ \mid y = \Sigma(x),\ \text{for some } x \in S\}$. The **aggregate endowment** $\Sigma(\omega)$ is denoted Ω.

An allocation $x \in X$ is **feasible for $(\mathbf{R}, \omega)$** if $\Sigma(x) \leqq \Omega$. For each $(R, \omega) \in \mathcal{E}$, its set of feasible allocations is denoted $F(R, \omega)$. An allocation $x \in X$ is **efficient for** (R, ω) if there is no allocation $x' \in F(R, \omega)$ such that, for each $i \in N$, $x'_i\ R_i\ x_i$, and for some $i \in N$, $x'_i\ P_i\ x_i$. For each $(R, \omega) \in \mathcal{E}$, its set of efficient allocations is denoted $P(R, \omega)$.

Given a preference profile $R \in \mathcal{R}^n$ and an allocation $x \in X$, the **upper contour set of $\boldsymbol{R}$ at $\boldsymbol{x}$** is $U(R, x) = \{x' \in X \mid x'\ R\ x\}$.[11] Similarly, $L(R, x) = \{x' \in X \mid x\ R\ x'\}$ is the **lower contour set of $\boldsymbol{R}$ at $\boldsymbol{x}$**.[12]

[7] The set $\mathbb{R}_+$ is the set of non-negative reals and the set $\mathbb{R}_{++}$ is the set of positive reals. Vector inequalities: $x, y \in \mathbb{R}^k$, $j = 1, ..., k$, $x \geqq y \iff x_j \geq y_j$. $x \geqq\!\!\!\!\!/\;\; y \iff x \geqq y$ and $x \neq y$. $x > y \iff x_j > y_j$.

[8] Given a preference relation $R \in \mathcal{R}$, we denote strict preference by P and indifference by I.

[9] For each $s \in \mathbb{R}^m$, and each $r \in \mathbb{R}$, $x > r$ denotes $x > (r_1, ..., r_m)$.

[10] In Section 5 we discuss relaxing this condition. We also discuss relaxing strict monotonicity of preferences.

[11] Given a preference profile R and a pair of allocations $x, x' \in X$, $x\ R\ x' \Leftrightarrow$ for each $i \in N$, $x_i\ R_i\ x'_i$. Similar notation is used for strict preference P and indifference I.

[12] If $n = 1$, the definitions correspond to the standard notions of the upper and lower contour sets

An allocation $x \in X$ is an **endowment-Pareto-indifferent allocation** if, for each $i \in N$, $x_i \; I_i \; \omega_i$. The set of all such allocations is denoted $I(R, \omega)$. Most endowment-Pareto-indifferent allocations are not feasible. An allocation $x \in X$, is a **reference allocation for** $(\mathbf{R}, \omega)$ if: (i) $x \in I(R, \omega)$, and (ii) $x \in F(R, \omega)$. The **set of reference allocations** is denoted $FI(R, \omega)$. For each $x \in FI(R, \omega)$, the vector $y = \Sigma(x)$ is a vector of **reference resources**. The **set of reference resources** is denoted $\Sigma FI(R, \omega)$.

Trading from the endowment profile to a *reference allocation* x leads to gains equal to the difference between the aggregate endowment and the resources at x. By considering the possibility of trading to each *reference allocation*, we obtain the **set of possible gains from trade:**[13]

$$G(R, \omega) = \{z \in \mathbb{R}^m \mid z = \Omega - \Sigma(x), \; x \in FI(R, \omega)\}.$$

The *set of possible gains from trade* is **symmetric** to the *set of reference resources*: $z \in G(R, \omega) \;\Leftrightarrow\; (\Omega - z) \in \Sigma FI(R, \omega)$. Geometrically, $\Sigma FI(R, \omega)$ is the intersection of two sets: (i) the set summation, over all agents, of their sets of endowment-Pareto-indifferent bundles, and (ii) the set of all vectors dominated by the aggregate endowment. For simplicity, we allow for **free-disposal**: if $y \in \Sigma FI(R, \omega)$, for each $y' \in \mathbb{R}^M$ with $y \leqq y' \leqq \Omega$, $y' \in \Sigma FI(R, \omega)$. Given free-disposal, condition (i) can be stated in terms of upper contour sets instead of sets of endowment-Pareto-indifferent bundles (see Figure 1).[14]

If the endowment profile of an economy is not an efficient allocation, the *set of possible gains from trade* contains more than one element. In order to measure the gains from trade in the economy, we select a representative vector from this set:

Definition 1. For each $(R, \omega) \in \mathcal{E}$, a **(vector-valued) metric** selects a vector $Q(R, \omega)$, such that, for each $(R', \omega') \in \mathcal{E}$:

(i) $Q(R, \omega) \in G(R, \omega)$, and

(ii) $G(R, \omega) = G(R', \omega') \Rightarrow Q(R, \omega) = Q(R', \omega')$.

We interpret a *metric* as selecting for each economy a representative vector from its *set of possible gains from trade*. Condition (i) states that there exists a *reference*

of a preference relation at a bundle.

[13]Throughout the paper, x denotes allocations, y denotes aggregate resources at allocations, and z denotes vectors of gains from trade.

[14]Given condition (ii), free-disposal is not necessary for the results, but it simplifies the proofs.

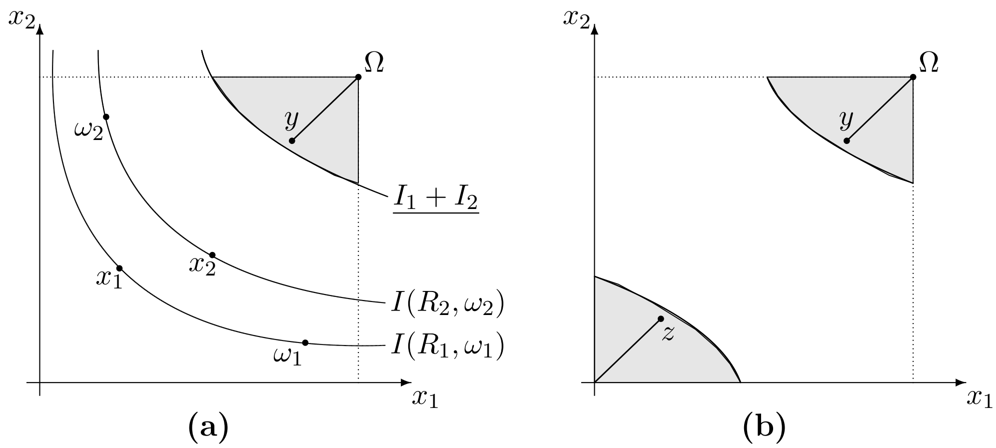

Figure 1: **Set of possible gains from trade.** (a) *Set of aggregate resources,* $\Sigma FI(R, \omega)$. Each vector y in the shaded area is a vector of *reference resources.* There exists $x \; I \; \omega$, with $\Sigma(x) = y$. Trading from ω to x leads to gains equal to $\Omega - y$. The curve $\underline{I_1 + I_2}$ is the lower boundary of the set $I(R_1, \omega_1) + I(R_2, \omega_2)$. (b) *Set of possible gains from trade,* $G(R, \omega)$. The shaded area in the bottom left is the *set of possible gains from trade.* It is *symmetric* to the *set of reference resources* (shaded area in the top right). Each $z \in G(R, \omega)$, is equal to the difference between Ω and a vector of *reference resources* y.

allocation leading to the vector selected by the *metric*. Condition (ii) states that the *metric* selects equal representative vectors in economies with equal *sets of possible gains from trade*.[15]

Definition 2. Given a *metric* Q, for each $(R, \omega) \in \mathcal{E}$, an allocation x is **Q-consistent** if: (i) $x \in FI(R, \omega)$, and (ii) $\Sigma(x) = \Omega - Q(R, \omega)$. The set of ***Q*-consistent allocations** is denoted $q(R, \omega)$.[16]

3 Measuring gains from trade

In this section, we introduce the "weighted-gains family" of *metrics*. We also introduce properties to test the behavior of *metrics*. Then, based on these properties, we provide a characterization of the weighted-gains family. We begin by introducing some mathematical concepts and preliminary results about the *set of possible gains from trade*.

[15] In Section 5 we discuss relaxing this assumption.

[16] If the upper contour set of R at ω is strictly convex, then $q(R, \omega)$ is a singleton.

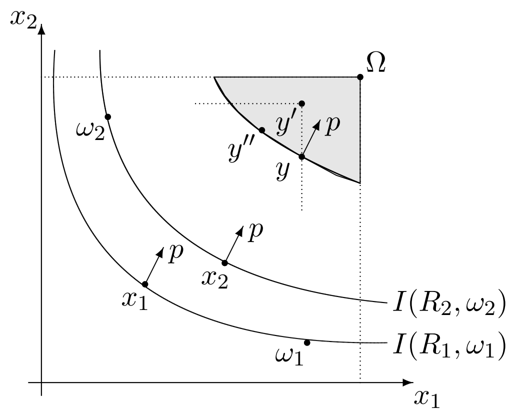

Figure 2: **The *set of possible gains from trade* is strictly comprehensive.** The pair of vectors y, y' are vectors of *reference resources*. The *reference allocation* $x \in FI(R, \omega)$, is such that, $x_1 + x_2 = y$. Taking y to be a minimal vector of *reference resources*, there exists p supporting $U(R_i, \omega_i)$ at x_i. Moreover, p supports $\Sigma FI(R, \omega)$ at y. Since preferences are strictly monotonic, p is strictly positive. Then, moving from y in the direction of y', we obtain an interior allocation. Thus, there exists $y'' \in \Sigma FI(R, \omega)$, such that, $y'' < y'$. By symmetry, $G(R, \omega)$ is strictly comprehensive.

3.1 Concepts and preliminary results

Given a set $S \subset \mathbb{R}^m_+$, $s \in S$ is a **maximal** element of S, if for each $s' \gneqq s$, $s' \notin S$. It is a **minimal** element of S, if for each $s' \lneqq s$, $s' \notin S$. A set $S \subset \mathbb{R}^m_+$ is **comprehensive** if for each $s \in S$, if $0 \leqq s' \leqq s$, $s' \in S$. A set $S \subset \mathbb{R}^m_+$ is **strictly comprehensive** if it is comprehensive, and for each pair $s, s' \in S$, with $s \gneqq s'$, there exists $s'' \in S$, such that $s'' > s'$. Let $S \in \mathbb{R}^m$, $s \in S$, and $p \in \mathbb{R}^m$, ***p* supports *S* at *s***, if for each $s' \in S$, $p \cdot s \leq p \cdot s'$.

Our first proposition states that *sets of possible gains from trade* satisfy some properties usually assumed for sets of feasible utility in the theory of bargaining (see Figure 2).

Proposition 1. *For each* $(R, \omega) \in \mathcal{E}$*,* $G(R, \omega)$ *is closed, convex, bounded above, and strictly comprehensive. Moreover, for each maximal element* $z \in G(R, \omega)$*, if* p *supports* $G(R, \omega)$ *at* z*, then,* $p < 0$*.*

Proof. First, we show that $G(R, \omega)$ is closed and convex. Let $(R, \omega) \in \mathcal{E}$. By free-disposal, $\Sigma FI(R, \omega) = \sum_{i \in N} U(R_i, \omega_i) \cap [0, \Omega]$.[17] Since preferences are continuous and convex, for each $i \in N$, $U(R_i, \omega_i)$ is closed and convex. Thus, $\Sigma FI(R, \omega)$ is closed and convex. By symmetry, so is $G(R, \omega)$.

[17]For each pair $x, x' \in \mathbb{R}^k$, with $x' \geqq x$, $[x, x'] \equiv \{x'' \in \mathbb{R}^k \mid x \leqq x'' \leqq x'\}$.

Now, we show that $G(R,\omega)$ is bounded above. Let $z \in G(R,\omega)$. Since $z \in G(R,\omega)$, there exists $x \in FI(R,\omega)$, such that, $z = \Omega - \Sigma(x)$. Since $x \in FI(R,\omega)$, for each $i \in N$, $x_i \geqq 0$. Thus, $\Sigma(x) \geqq 0$, and $\Omega \geqq z$.

Now, we show that for each maximal element z of $G(R,\omega)$, if p supports $G(R,\omega)$ at z, then, $p < 0$. Let $z \in G(R,\omega)$ and $p \in \mathbb{R}^m$ support $G(R,\omega)$ at z. Let $y = \Omega - z$. Since z is a maximal element of $G(R,\omega)$, by symmetry, y is a minimal element of $\Sigma FI(R\Omega)$. Thus, $-p$ supports $\Sigma FI(R,\omega)$ at y. By definition of y, there exists $x \in FI(R,\omega)$ such that $y = \Sigma(x)$. Moreover, since y is a minimal element of $\Sigma FI(R,\omega)$, for each $i \in N$, $-p$ supports R_i at x_i. By strict monotonicity of preferences, $-p > 0$.[18]

Finally, we show that $G(R,\omega)$ is strictly comprehensive. Comprehensiveness of $G(R,\omega)$ follows directly form symmetry and free-disposal. Let $z \in G(R,\omega)$ and $z' \lneqq z$. Without loss of generality, assume z is a maximal element of $G(R,\omega)$. By the previous step, each supporting price of $G(R,\omega)$ at z is strictly negative. Then, for each $\alpha \in (0,1)$, $\alpha z + (1-\alpha)z'$ is an interior element of $G(R,\omega)$. Thus, there exists $z'' \in G(R,\omega)$ with $z' < z''$. □

The next proposition is a converse statement of Proposition 1. Each set satisfying the properties stated in Proposition 1 coincides with the *set of possible gains from trade* of some economy (see Figure 3).

Proposition 2. *Let $Z \subset \mathbb{R}^m$ be closed, convex, bounded above, strictly comprehensive, and such that for each maximal element $\bar{z} \in G(R,\omega)$, if p supports $G(R,\omega)$ at $\bar{z}$, then, $p < 0$. Then, there exists $(R,\omega) \in \mathcal{E}$ such that $G(R,\omega) = Z$.*

Proof. We construct such an economy for n even. For n odd a similar construction can be obtained. Without loss of generality assume $Z \neq \{0\}$.[19] We construct an economy where all agents have equal preferences.

Step 1: Specifying the aggregate endowment. For each $k \in M$, let $\bar{z}_k = \max_{\{z \in Z\}} z_k$ and $\bar{z} = (\bar{z}_k)_{k \in M}$. Let $\Omega_1 = \bar{z}_1 + \bar{z}_2$, $\Omega_2 = \bar{z}_1 + \bar{z}_2$, $\Omega_{N\setminus\{1,2\}} = \bar{z}_{N\setminus\{1,2\}} + 1$.

Step 2: Providing restrictions on preferences. Let $Y = \Omega - Z$, $\bar{Y}$ be the set of minimal elements of Y, and $\bar{X} = \frac{1}{n}\bar{Y} = \{x \in \mathbb{R}^m \mid x = \frac{1}{n}y,\ y \in \bar{Y}\}$.

Let $\bar{x}^1 = \left(\frac{\bar{z}_2}{n}, \frac{\Omega_2}{n}, \frac{\Omega_{N\setminus\{1,2\}}}{n}\right)$ and $\bar{x}^2 = \left(\frac{\Omega_1}{n}, \frac{\bar{z}_1}{n}, \frac{\Omega_{N\setminus\{1,2\}}}{n}\right)$. Note that, for each $k \in \{1,2\}$, $\bar{x}^k \in \bar{X}$ and $0 < \bar{x}^k \lneqq \frac{1}{n}\Omega$. Let $\varepsilon > 0$, and:

$$\bar{x}'^1 = \left(\frac{\bar{z}_2}{n} - \frac{\varepsilon}{n}, \frac{\bar{z}_1}{n} + \frac{2\bar{z}_2}{n} + \frac{\bar{z}_2}{\bar{z}_1}\frac{\varepsilon}{n}, \frac{\Omega_3}{n}, ..., \frac{\Omega_m}{n}\right),$$

[18]Since $x \in FI(R,\omega)$, and for each $i \in N$, $\omega_i > 0$, by boundary aversion, $x_i > 0$.

[19]If $Z = \{0\}$, let $(R,\omega) \in \mathcal{E}$ such that ω is an efficient allocation. Then, $G(R,\omega) = \{0\}$.

$$\bar{x}'^2 = \left(\frac{2\bar{z}_1}{n} + \frac{\bar{z}_2}{n} + \frac{\varepsilon}{n}, \frac{\bar{z}_1}{n} - \frac{\bar{z}_2}{\bar{z}_1}\frac{\varepsilon}{n}, \frac{\Omega_3}{n}, ..., \frac{\Omega_m}{n}\right).$$

Note that $\bar{x}'^1$ and $\bar{x}'^2$ are symmetric with respect to $\frac{\Omega}{2}$.[20] Consider the following conditions on a preference relation R_i:

(i) For each $x, x' \in \bar{X}$, $x \; I_i \; x'$.

(ii) For each $\alpha \in [0,1]$, $(\alpha \bar{x}^1 + (1-\alpha)\bar{x}'^1) \; I_i \; \bar{x}^1$.

(iii) For each $\alpha \in [0,1]$, $(\alpha \bar{x}^2 + (1-\alpha)\bar{x}'^2) \; I_i \; \bar{x}^2$.

Step 3: Specifying preferences. For ε small enough, there exists a homothetic preference relation $\bar{R}_i \in \mathcal{R}$, consistent with conditions (i)-(iii). Let $\bar{R} = (\bar{R}_i)_{i \in N}$.

Step 4: Specifying the endowment profile. Let $N_1 = \{1, ..., \frac{n}{2}\}$ and $N_2 = N \setminus N_1$. Let $\bar{\omega} = (\bar{x}'^1_{N_1}, \bar{x}'^2_{N_2})$. Then, $\sum_{i \in N} \bar{\omega}_i = \Omega$.

Let z be a maximal element of Z. We show that z is a maximal element of $G(\bar{R}, \bar{\omega})$. Let $y = \Omega - z$. Then, $y \in \bar{Y}$. For each $i \in N$, let $x_i = \frac{1}{n}y$. Then, $x_i \in \bar{X}$ and, by conditions (i)-(iii), $x_i \; I_i \; \omega_i$. Thus, $y \in \Sigma FI(R, \omega)$. Since all agents have the same preferences, at the common bundle $x_i = \frac{1}{n}y$ there exist a price vector p supporting, for each $i \in N$, $U(R_i, x_i)$. Hence, y is a minimal element of $\Sigma FI(R, \omega)$. By symmetry, z is a maximal element of $G(R, \omega)$. Since both Z and $G(R, \omega)$ are strictly comprehensive, and their maximal elements coincide, $Z = G(R, \omega)$. □

By Propositions 1 and 2, the union over all economies of their *sets of possible gains from trade* is equal to the set of all closed, convex, bounded above, strictly comprehensive sets with strictly negative supporting prices. For bargaining theory, the "egalitarian solution" is well-behaved on this domain of sets. This solution selects the maximal profile of equal utilities. For the problem of measuring gains from trade, some goods may be deemed more important than others, and equal gains of each good may not be desirable. Thus, we introduce a family of "weighted-gains metrics", each member of this family satisfying most of the desirable properties of the egalitarian solution, but allow asymmetric treatment across goods. To allow for asymmetries across goods we can assign a weight to each good, and measure gains from trade by the largest vector proportional to this vector of weights. (see Figure 4).

Definition 3. For each $\alpha \in \mathbb{R}^M_+ \setminus \{0\}$ and each $(R, \omega) \in \mathcal{E}$, the α**-weighted-gains metric** selects the vector, $Q^\alpha(R, \omega) = \bar{\lambda}\alpha$, where $\bar{\lambda} \in \mathbb{R}$ is such that, $\bar{\lambda}\alpha$ is a maximal element of $G(R, \omega)$.

[20] For half the population each agent's endowment will be set equal to $\bar{x}'^1$, and for the other half will be set equal to $\bar{x}'^2$. For n odd, an appropriate change is required so that $\frac{n+1}{2}\bar{x}'^1 + \frac{n-1}{2}\bar{x}'^2 = \frac{\Omega}{n}$.

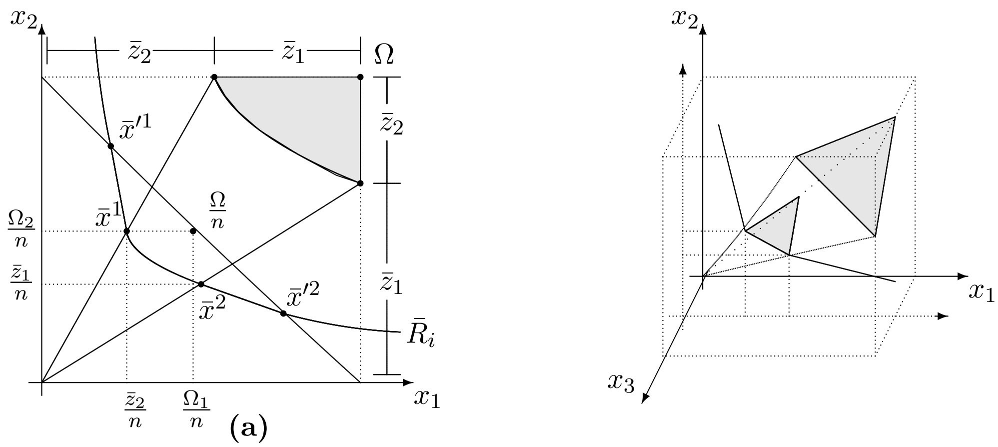

Figure 3: **Assigning an economy to each strictly comprehensive set.** *(a) An economy with two goods.* Let Z be a strictly comprehensive set. Define Ω to dominate each vector in Z. By symmetry we work with the set $Y = \Omega - Z$ (shaded area). $\bar{Y}$ is its set of minimal elements (lower boundary of the shaded area). Perform a homogeneous reduction of $\bar{Y}$ of scale $\frac{1}{n}$ (segment joining $\bar{x}^1$ and $\bar{x}^2$). Define $\bar{x}'^1$ to lie on the line with slope $-\frac{\bar{z}_2}{\bar{z}_1}$ passing through $\frac{\Omega}{n}$, such that it preserves convexity of preferences (segment joining $\bar{x}^1$ and $\bar{x}'^1$). Define $\bar{x}'^2$ to be symmetric to $\bar{x}'^1$ with respect to $\frac{\Omega}{n}$ (segment joining $\bar{x}^2$ and $\bar{x}'^2$). Continue the indifference curve preserving convexity, and define homothetic preferences consistent with this indifference curve. Set ω so that half the agents have endowments equal to $\bar{x}'^1$, and the other half equal to $\bar{x}'^2$. *(b) An economy with 3 goods.* Perform the homogeneous reduction of $\bar{Y}$. Set $x_3 = \frac{\Omega_3}{n}$. Follow the same procedure as for two goods.

3.2 Properties of metrics

Now, we introduce some properties of *metrics*. These properties are used as test of good behavior. Given the requirement that *metrics* select equal gains in economies with equal *sets of possible gains from trade*, we state some properties in terms of *sets of possible gains from trade*, and not in terms of the primitives of the model.

Metrics select for each economy a vector representative of its *set of possible gains from trade*. This vector is interpreted as representing the size of this set. Hence, it should select a maximal vector. The first property states that no *reference allocation* leads to larger gains than the vector selected by the *metric*:

Maximality: For each $(R, \omega) \in \mathcal{E}$ and each $y \in \mathbb{R}^m_+$, if $y \gneqq Q(R, \omega)$, then $y \notin G(R, \omega)$.

By definition, each member of the *weighted-gains family* of *metrics* satisfies *maximality*. Since *metrics* represent the size of the *set of possible gains from trade*, monotonicity with respect to set inclusion is a natural requirement.[21] The next prop-

[21] *Monotonicity* plays an important role for the application to fair allocation. It implies a sufficient condition for efficiency of the fair allocation we propose in Section 5. But it is not a necessary condition.

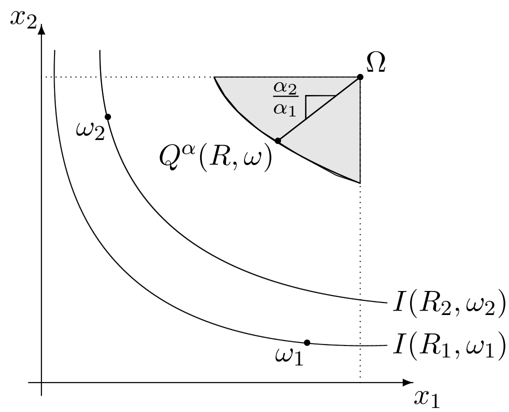

Figure 4: **The α-weighted-gains metric.** For each economy, the *α-weighted-gains metric* measures gains from trade proportional to α.

erty states that *metrics* select larger gains from trade in economies with larger *sets of possible gains from trade*:

Monotonicity: For each pair $(R,\omega),(R',\omega') \in \mathcal{E}$, if $G(R,\omega) \subset G(R',\omega')$, then $Q(R,\omega) \leqq Q(R',\omega')$.

A member of the *weighted-gains family* measures gains form trade in each pair of economies by the largest vector proportional to the same vector of weights. The vector so obtained is larger in an economy with a larger *set of possible gains from trade*. Thus, each member of the family satisfies *monotonicity*. The next property states that a homogeneous expansion of the *set of possible gains from trade* leads to the same homogeneous expansion of the vector selected by the *metric*. For low consumption levels, some goods may be more important than others and, as consumption increases, the relative importance of goods may change. But, since *sets of possible gains from trade* contain no information about the quantities of goods consumed, we require the vector selected by the *metric* to vary accordingly to homogeneous changes in the *sets of possible gains from trade*.[22]

Homogeneity: For each pair $(R,\omega),(R',\omega') \in \mathcal{E}$ and each $\lambda \in \mathbb{R}_+$, if $G(R,\omega) = \lambda G(R',\omega')$, then $Q(R,\omega) = \lambda Q(R',\omega')$.

Each member of the *weighted-gains family* measures gains form trade in each pair of economies proportionally to the same vector of weights. Thus, for each pair of economies, the vectors selected by the *metric* are homogeneous transformation of

[22]For the application to fair allocation all results hold without *homogeneity*. The characterization of the *weighted-gains family* depends on *homogeneity* but the family of *metrics* satisfying *maximality* and *monotonicity* is known in bargaining theory (Thomson 2004).

each other. By *maximality*, if the *sets of possible gains from trade* of the economies are homogeneous transformations of each other, then, the vectors selected by the *metric* are homogeneous transformations of each other of the same scale. Thus, each member of the family satisfies *homogeneity*.

3.3 A characterization of the weighted-gains family

As noted above, each member of the *weighted-gains family* satisfies *maximality*, *monotonicity*, and *homogeneity*. Next, we show that the converse statement is also true (see Figure 5).[23]

Theorem 1. *A metric Q satisfies maximality, monotonicity, and homogeneity, if and only if there exists $\alpha \in \mathbb{R}^m_+ \setminus \{0\}$ such that $Q = Q^\alpha$.*

Proof. We noted above that for each $\alpha \in \mathbb{R}^m_+ \setminus \{0\}$, Q^α satisfies the three properties. We show the converse.

Step 0: Obtaining α. Let $\bar{Z} = \{z \in \mathbb{R}^m_+ \mid \sum_{j \in m} z_j \leqq 1\}$. Then, $\bar{Z}$ is closed, convex, bounded above, strictly comprehensive, and at each maximal element z, if p supports Z at z, then, $p < 0$. By Proposition 2, there exists $(\bar{R}, \bar{\omega}) \in \mathcal{E}$ with $G(\bar{R}, \bar{\omega}) = \bar{Z}$. Let $\alpha = Q(\bar{R}, \bar{\omega})$. By *maximality*, $\alpha \in \mathbb{R}^m_+ \setminus \{0\}$.

Let $(R, \omega) \in \mathcal{E}$, $Z = G(R, \omega)$, and $z^* = Q^\alpha(R, \omega)$. We need to show that $Q(R, \omega) = z^*$.

Step 1: Measuring gains in the homogeneous expansion of $\bar{Z}$. Let $k = \sum_{j \in M} z^*_j$, and $\bar{Z}' = \{z \in \mathbb{R}^m_+ \mid \sum_{j \in m} z_j \leqq k\}$. By Proposition 2, there exists $(\bar{R}', \bar{\omega}') \in \mathcal{E}$ with $G(\bar{R}', \bar{\omega}') = \bar{Z}'$. Since $G(\bar{R}', \bar{\omega}') = kG(\bar{R}, \bar{\omega})$, by *homogeneity*, $Q(\bar{R}', \bar{\omega}') = k\alpha = Q^\alpha(\bar{R}', \bar{\omega}') = z^*$.

Step 2: Measuring gains in the intersection of Z and $\bar{Z}'$. Let $Z' = Z \cap \bar{Z}'$. By Proposition 1, Z is closed, convex, strictly comprehensive and for each maximal vector, supporting prices are negative. By definition, so is $\bar{Z}'$. Then, so is their intersection Z'. Moreover, $z^* \in Z'$. By Proposition 2, there exists $(R', \omega') \in \mathcal{E}$ with $G(R', \omega') = Z'$. We show that $Q(R', \omega') = z^*$. Let $Q(R, \omega) = z'$. Since $G(\bar{R}', \bar{\omega}') = \bar{Z}'$ and $Q(\bar{R}', \bar{\omega}') = z^*$, by *maximality*, z^* is maximal for $\bar{Z}'$. By definition of Z', z^* is also maximal for Z'. Since $G(R', \omega') = Z'$ and $z' = Q(R', \omega')$, by *maximality*, z' is maximal for Z'. Hence, both z^* and z' are maximal for Z'. By *monotonicity*, $z^* = Q(\bar{R}', \bar{\omega}') \geqq Q(R', \omega') = z'$. Hence, $z' = z^* = Q(R', \omega')$.

[23] For bargaining theory, a similar result holds if we replace *maximality* by a "weak maximality" requirement (Kalai 1977).

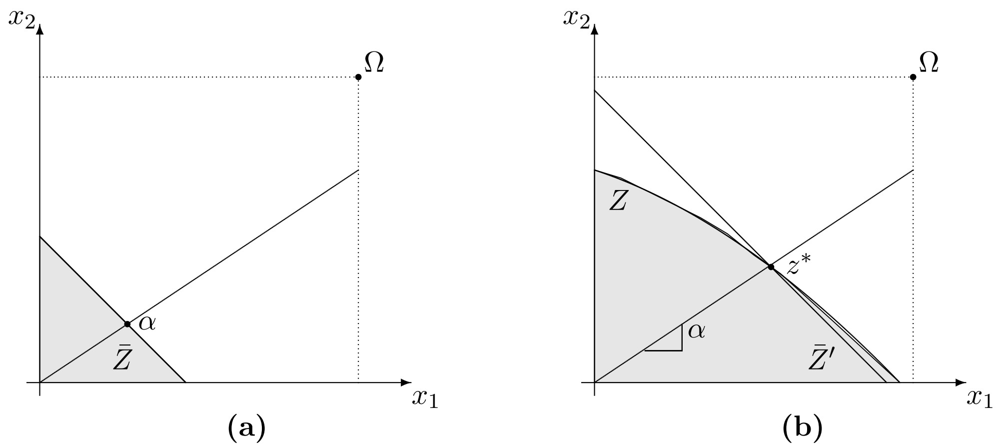

Figure 5: **Characterization of the *weighted-gains family*.** (a) *Obtaining* α. The set $\bar{Z}$ is the *set of possible gains from trade* of some economy $(\bar{R}, \bar{\omega})$. Let $\alpha = Q(R, \omega)$. (b) *Showing that* $Q \equiv Q^\alpha$. Let Z be the *set of possible gains from trade* of some economy (R, ω) and $z^* = Q^\alpha(R, \omega)$. Define $\bar{Z}'$ to be a homogeneous expansion of $\bar{Z}$ passing through z^*. $\bar{Z}'$ is the *set of possible gains from trade* of some economy $(\bar{R}', \bar{\omega}')$. By *homogeneity*, $z^* = Q(\bar{R}', \bar{\omega}')$. The intersection of Z and $\bar{Z}'$ is the *set of possible gains from trade* of some economy (R', ω'). By *monotonicity*, going from $\bar{Z}'$ to the intersection of Z and $\bar{Z}'$ implies $z^* = Q(R'\omega')$. Then, by *monotonicity*, going from the intersection of Z and $\bar{Z}'$ to Z implies $z^* = Q(R, \omega)$.

Step 3: Measuring gains in the original set Z. We show that $Q(R, \omega) = z^*$. Let $z = Q(R, \omega)$. Since $G(R, \omega) = Z$ and $z^* = Q^\alpha(R, \omega)$, by *maximality*, z^* is maximal for Z. As noted above, z also is maximal for Z'. Since $G(R, \omega) = Z$ and $z = Q(R', \omega')$, by *maximality*, z is maximal for Z. Hence, both z^* and z are maximal for Z. By *monotonicity*, $z = Q(R, \omega) \geqq Q(R', \omega') = z^*$. Hence, $z^* = z = Q(R, \omega)$. □

For the independence of the properties well-known solutions in bargaining theory provide examples. We refer to Thomson (2004) for a detailed treatment.

4 An application to fair allocation

Once we have a measure of the gains from trade in an economy, how can we distribute them fairly? An "allocation rule" recommends for each economy a set of feasible allocations. We look for allocations rules that distribute gains from trade fairly. First, we propose a method to determine the contribution of each agent to the gains from trade. Then, we declare an allocation "fair" if each agent obtains her contribution to the gains from trade. Finally, we propose an allocation rule which assigns to each agent her contribution to the gains from trade. This rule is not efficient, but we show that a recursive procedure distributes gains from trade fairly at each step and defines an efficient rule.

4.1 Contributions to gains from trade

In order to determine each agent's contribution to the gains from trade, we propose to use the solution concept of the theory of cooperative games known as the Shapley-value, using its interpretation as rewarding agents as a function of their "marginal contributions" to all subgroups. We measure each agent's contribution to the gains from trade as the "marginal gains" in each subpopulation.

First, we generalize the definition of the *weighted-gains family* to allow for variable populations. For each subpopulation $N' \subset N$ and each economy $(R, \omega) \in \mathcal{E}$, the α-*weighted-gains metric* measures gains from trade of the subeconomy $(R_{N'}, \omega_{N'})$ by the largest vector $z \in G(R_{N'}, \omega_{N'})$ proportional to α.

Definition 4. For each $\alpha \in \mathbb{R}^M_+ \setminus \{0\}$, each $N' \subset N$, and each $(R, \omega) \in \mathcal{E}$, the **$\alpha$-weighted-gains metric** selects the vector of gains from trade in the economy $(R_{N'}, \omega_{N'})$, $Q^\alpha(R_{N'}, \omega_{N'}) = \bar{\lambda}\alpha$, where $\bar{\lambda} \in \mathbb{R}$ is such that $\bar{\lambda}\alpha$ is a maximal element of $G(R_{N'}, \omega_{N'})$.

Now, imagine agents arriving one at a time and calculate, for each agent, the difference between: (i) the gains in the economy consisting of her and all agents preceding her, and (ii) the gains in the economy consisting of all agents preceding her:

Definition 5. For each $\alpha \in \mathbb{R}^m_+ \setminus \{0\}$, each permutation of the population $\pi \in \Pi$, each $i \in N$, and each $(R, \omega) \in \mathcal{E}$, **i's π -contribution to the gains from trade** is:

$$C_i^{\alpha,\pi}(R, \omega) = Q^\alpha(R_{\overline{\pi}(i)}, \omega_{\overline{\pi}(i)}) - Q^\alpha(R_{\underline{\pi}(i)}, \omega_{\underline{\pi}(i)}),$$

where $\overline{\pi}(i) = \{j \in N \mid \pi(j) \leq \pi(i)\}$, and $\underline{\pi}(i) = \{j \in N \mid \pi(j) < \pi(i)\}$.

Different orders of arrival lead to different contributions. We take the average of each agent's contribution to the gains from trade over all possible orders of arrival, under the premise that all orders are equally likely, as a measure of her contribution to the gains from trade:

Definition 6. For each $\alpha \in \mathbb{R}^m_+ \setminus \{0\}$, each $i \in N$, and each $(R, \omega) \in \mathcal{E}$, **i's contribution to the gains from trade** is:

$$C_i^\alpha(R, \omega) = \frac{1}{n!} \sum_{\pi \in \Pi} C_i^{\alpha,\pi}(R, \omega).$$

The **profile of marginal contributions** is:

$$C^{\alpha}(R,\omega) = (C_i^{\alpha}(R,\omega))_{i\in N}.$$

The arrival of a new agent to an economy leads to a larger *set of possible gains from trade*. Thus, each agent's *contribution to the gains from trade* is positive:

Proposition 3. *For each $\alpha \in \mathbb{R}^m_+ \setminus \{0\}$ and each $(R,\omega) \in \mathcal{E}$, $C^{\alpha}(R,\omega) \geqq 0$.*

Proof. Let $\pi \in \Pi$, $i \in N$, and $(R,\omega) \in \mathcal{E}$. We claim that $G(R_{\underline{\pi}(i)}, \omega_{\underline{\pi}(i)}) \subset G(R_{\overline{\pi}(i)}, \omega_{\overline{\pi}(i)})$.

Let $z \in G(R_{\underline{\pi}(i)}, \omega_{\underline{\pi}(i)})$, and $x \in FI(R_{\underline{\pi}(i)}, \omega_{\underline{\pi}(i)})$ such that $z = \Omega_{\underline{\pi}(i)} - \Sigma(x)$. Then, $(x, \omega_i) \in FI(R_{\overline{\pi}(i)}, \omega_{\overline{\pi}(i)})$, and $\Omega_{\overline{\pi}(i)} - \Sigma(x,\omega_i) = z$. Thus, $z \in G(R_{\overline{\pi}(i)}, \omega_{\overline{\pi}(i)})$.

By *monotonicity*, $Q^{\alpha}(R_{\overline{\pi}(i)}, \omega_{\overline{\pi}(i)}) \geqq Q^{\alpha}(R_{\underline{\pi}(i)}, \omega_{\underline{\pi}(i)})$. Thus, $C_i^{\alpha,\pi}(R,\omega) \geqq 0$.

Since $\pi \in \Pi$ is arbitrary, $C_i^{\alpha}(R,\omega) = \frac{1}{n!}\sum_{\pi\in\Pi} C_i^{\alpha,\pi}(R,\omega) \geqq 0$. Since $i \in N$ is arbitrary, $C^{\alpha}(R,\omega) \geqq 0$. □

4.2 Allocation of gains from trade

We apply our definitions to obtain a fair allocation. An allocation rule recommends for each economy a set of feasible allocations:

Definition 7. For each economy $(R,\omega) \in \mathcal{E}$, an **allocation rule**, φ, selects a set of feasible allocations $\varphi(R,\omega) \subset F(R,\omega)$.

An allocation is fair if each agent receives her contribution to the gains from trade. We assign to each agent her contribution to the gains from trade by assigning her the bundle obtained as the sum of: (i) her bundle in a *reference allocation* leading to gains equal to the vector selected by the *metric*, and (ii) her *contribution to the gains from trade*.

For some economies, there exist several *reference allocations* leading to gains equal to the vector selected by the metric. The allocation rule we propose recommends all allocations that can be obtained in the manner described above:[24]

Definition 8. For each $\alpha \in \mathbb{R}^m_+ \setminus \{0\}$ and each $(R,\omega) \in \mathcal{E}$, the α-**weighted-gains allocation rule** recommends the set of allocations $\varphi^{\alpha}(R,\omega) = q^{\alpha}(R,\omega) + C^{\alpha}(R,\omega)$.

For each $\alpha \in \mathbb{R}^m_+ \setminus \{0\}$, there exist economies for which the allocations recommended by the *α-weighted-gains allocation rule* are inefficient. Then, any allocation

[24] Recall that when preferences are strictly convex, for each economy, there exists a unique reference allocation.

Pareto-dominating one of the recommended allocations assigns to each agent at least her contribution to the gains from trade. We can interpret the *α-weighted-gains allocation rule* as providing a lower bound on the welfare that each agent achieves:

Definition 9. For each $\alpha \in \mathbb{R}_+ \setminus \{0\}$ and each $(R, \omega) \in \mathcal{E}$, an allocation $x \in F(R, \omega)$ **satisfies the *α-weighted-gains lower bound***, if there exists $x' \in \varphi^\alpha(R, \omega)$ such that $x \; R \; x'$. The set of allocations satisfying the *α-weighted-gains lower bound* is denoted $\bar{\varphi}^\alpha(R, \omega)$.

If for some economy, an allocation recommended by a *weighted-gains allocation rule* is not efficient, gains from trade in the resulting economy are positive. We distribute these gains according to the same rule:

Definition 10. For each $\alpha \in \mathbb{R}^m_+ \setminus \{0\}$ and each $(R, \omega) \in \mathcal{E}$, the **$\alpha^2$-weighted-gains allocation rule** recommends the set of allocations:

$$\varphi^{\alpha^2}(R, \omega) = \bigcup_{x \in \varphi^\alpha(R,\omega)} q^\alpha(R, x) + C^\alpha(R, x).$$

Definition 11. For each $\alpha \in \mathbb{R}_+ \setminus \{0\}$ and each $(R, \omega) \in \mathcal{E}$, an allocation $x \in F(R, \omega)$ **satisfies the *α^2-weighted-gains lower bound***, if there exists $x' \in \varphi^{\alpha^2}(R, \omega)$ such that $x \; R \; x'$. The set of allocations satisfying the *α^2-weighted-gains lower bound* is denoted $\bar{\varphi}^{\alpha^2}(R, \omega)$.

We proceed recursively, distributing gains from trade fairly at each step:

Definition 12. For each $\alpha \in \mathbb{R}^m_+ \setminus \{0\}$, each $k \in \mathbb{N}$, and each $(R, \omega) \in \mathcal{E}$, the **$\alpha^{\mathbf{k}}$-weighted-gains allocation rule** recommends the set of allocations:

$$\varphi^{\alpha^k}(R, \omega) = \bigcup_{x \in \varphi^{\alpha^{k-1}}(R,\omega)} q^\alpha(R, x) + C^\alpha(R, x).$$

Definition 13. For each $\alpha \in \mathbb{R}_+ \setminus \{0\}$, each $k \in \mathbb{N}$, and each $(R, \omega) \in \mathcal{E}$, a feasible allocation $x \in F(R, \omega)$ **satisfies the *$\alpha^{\mathbf{k}}$-weighted-gains lower bound***, if there exists $x' \in \varphi^{\alpha^k}(R, \omega)$ such that $x \; R \; x'$. The set of allocations satisfying the *α^k-weighted-gains lower bound* is denoted $\bar{\varphi}^{\alpha^k}(R, \omega)$.

The next lemma shows that the sequence of sets of allocations satisfying the lower bounds is a sequence of nested sets:

Lemma 1. *For each $\alpha \in \mathbb{R}_+ \setminus \{0\}$, each $k \in \mathbb{N}$, and each $(R, \omega) \in \mathcal{E}$, $\overline{\varphi}^{\alpha^{k+1}}(R, \omega) \subset \overline{\varphi}^{\alpha^k}(R, \omega)$.*

Proof. Let $x \in \overline{\varphi}^{\alpha^{k+1}}(R,\omega)$. By definition of $\overline{\varphi}^{\alpha^{k+1}}$, there exists $x' \in \varphi^{\alpha^{k+1}}(R,\omega)$, such that $x \; R \; x'$. By definition of $\varphi^{\alpha^{k+1}}$, there exists $x'' \in \varphi^{\alpha^{k}}(R,\omega)$ such that $x' \in q^{\alpha}(R,x'') + C^{\alpha}(R,x'')$. Let $\hat{x} \in q^{\alpha}(R,x'')$ be such that $x' = \hat{x} + C^{\alpha}(R,x'')$. Since $\hat{x} \in q^{\alpha}(R,x'')$, $\hat{x} \; I \; x''$. Moreover, $C^{\alpha}(R,x'') \geqq 0$. Then, by strict monotonicity, $x' \; R \; \hat{x}$. By transitivity, $x \; R \; x''$. Thus, $x \in \overline{\varphi}^{\alpha^{k}}(R,\omega)$. □

As the next proposition shows, the limit of the nested sequence defined by the lower bounds is non-empty.

Proposition 4. *For each $\alpha \in \mathbb{R}_+ \setminus \{0\}$ and each $(R,\omega) \in \mathcal{E}$, let $\phi^{\alpha}(R,\omega) = \bigcap_{k\in\mathbb{N}} \overline{\varphi}^{\alpha^{k}}(R,\omega)$. Then, $\phi(R,\omega) \neq \emptyset$.*

Proof. Since preferences are continuous, for each $k \in \mathbb{N}$, $\overline{\varphi}^{\alpha^{k}}(R,\omega)$ is a closed set. Since $F(R,\omega)$ is bounded, $\overline{\varphi}^{\alpha^{k}}(R,\omega) \subset F(R,\omega)$ is also bounded. Moreover, for each $k \in \mathbb{N}$, $\varphi^{\alpha^{k}}(R,\omega) \subset \overline{\varphi}^{\alpha^{k}}(R,\omega)$. Thus, $\overline{\varphi}^{\alpha^{k}}(R,\omega) \neq \emptyset$. Hence, by Lemma 1, $\{\overline{\varphi}^{\alpha^{k}}(R,\omega)\}_{k\in\mathbb{N}}$ defines a sequence of non-empty, compact, and nested sets. Thus, $\bigcap_{k\in\mathbb{N}} \overline{\varphi}^{\alpha^{k}}(R,\omega) \neq \emptyset$. □

The limit of this sequence defines the "recursive weighted-gains allocation rule":

Definition 14. For each $\alpha \in \mathbb{R}^m_+ \setminus \{0\}$ and each $(R,\omega) \in \mathcal{E}$, the α-**recursive-weighted-gains allocation rule** recommends the set of allocations:

$$\phi^{\alpha}(R,\omega) = \bigcap_{k\in\mathbb{N}} \overline{\varphi}^{\alpha^{k}}(R,\omega).$$

For each vector of weights the *recursive-weighted-gains allocation rules* is efficient (see Figure 6):

Theorem 2. *Let $\alpha \in \mathbb{R}^m_+ \setminus \emptyset$ and $(R,\omega) \in \mathcal{E}$, then, each $x \in \phi^{\alpha}(R,\omega)$ is efficient.*

Proof. By contradiction, assume there exists $\bar{x} \in \phi^{\alpha}(R,\omega)$ and $\bar{x}' \in F(R,\omega)$, with $\bar{x}' \; R \; \bar{x}$, and for some $i \in N$, $\bar{x}'_i \; P_i \; \bar{x}_i$.

Then, there exists a sequence $\{x^k\}_{k\in\mathbb{N}}$ with:

(i) for each $k \in \mathbb{N}$, $x^{k+1} \in (q^{\alpha}(R,x^k) + C^{\alpha}(R,x^k))$, and

(ii) for each $k \in \mathbb{N}$, $\bar{x} \; R \; x^k$.

Let $\{(x^k, q^k, c^k))\}_{k\in\mathbb{N}}$ be a sequence satisfying condition (i), such that, for each $k \in \mathbb{N}$, $x^{k+1} = q^k + c^k$. Then for each $k \in \mathbb{N}$, we have $x^k \in F(R,\omega)$, $q^k \in F(R,\omega)$, and $c^k \in [0,\Omega]^n$. Then, since $\{(x^k, q^k, c^k))\}_{k\in\mathbb{N}}$ is a sequence in a compact set, it has

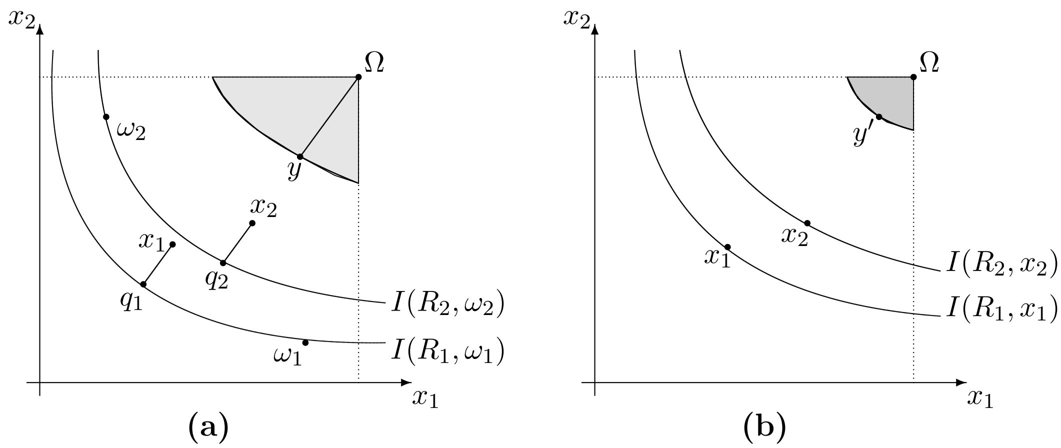

Figure 6: **The α-weighted-gains allocation rule.** (a) *Assigning contributions.* The vector y is the vector of gains from trade selected in the economy (R, ω). The allocation (q_1, q_2) is the *reference allocation* leading to gains $\Omega - y$. Assigning to each agent her contributions leads to the allocation (x_1, x_2). Since the sum of contributions is equal to the gains from trade, $x_1 + x_2 = \Omega$. (b) *The resulting economy.* The allocation (x_1, x_2) is inefficient. The vector y' is the vector of gains from trade selected in the economy (R, x). Since contributions are positive, each agent's welfare increases from ω to x. Hence, the *set of possible gains from trade* of the economy (R, x) is smaller than the *set of possible gains from trade* of the economy (R, ω). Repeated assignment of gains from trade leads to an economy with a *set of possible gains from trade* equal to $\{0\}$.

a convergent subsequence. Without loss of generality, we assume that the sequence itself is convergent.

Let $(x^*, q^*, c^*) = lim_{k\to\infty}(x^k, q^k, c^k)$. We claim that: $c^* = 0$. By Proposition 3, for each $k \in \mathbb{N}$, $c^k \geqq 0$. Thus, $c^* \geqq 0$. Moreover, for each $k \in \mathbb{N}$, $x^k \; I \; q^k$, thus, by continuity of preferences, $x^* \; I \; q^*$. By definition, $x^* = lim_{k\to\infty} q^k + c^k = q^* + c^*$. Then, by strict monotonicity of preferences, $c^* = 0$.

By definition, for each $k \in \mathbb{N}$, $Q^\alpha(R, x^k) = \sum_{i\in N} c_i^k$. By the previous step, $Q^\alpha(R, x^k) \to 0$. By *maximality* of Q, 0 is maximal for $lim_{k\to\infty} G(R, x^{l(k)})$. Then, by continuity of preferences, 0 is a maximal vector of $G(R, x^*)$. By Proposition 1, $G(R, x^*) = 0$. Hence, $x^* \in P(R, x^*)$. Since $\Sigma(x^*) = \Omega$, $x^* \in P(R, \omega)$.

By condition (ii), for each $k \in \mathbb{N}$, $\bar{x} \; R \; x^k$. Then, by continuity of preferences, $\bar{x} \; R \; x^*$. By the assumption that $\bar{x} \notin P(R, \omega)$, $x^* \notin P(R, \omega)$. This is a contradiction with the previous step. Hence, if $x \in \phi^\alpha(R, \omega)$, then $x \in P(R, \omega)$. □

The proof of Theorem 2 relies on *maximality* and *monotonicity* of the *metric*. In order to show that the gains from trade converge to zero, we used the fact that contributions are non-negative. This property of contributions relies on *monotonicity* of the *metric*. We conjecture that dropping *maximality* and requiring that at each step, each agent's welfare should increase is sufficient for Theorem 2 to hold.[25]

5 Conclusions

We proposed a method to measure gains from trade. We avoided interpersonal comparisons of welfare by defining gains in terms of quantities of goods. To do so, we introduced the notion of a *metric*. A *metric* measures gains from trade by a vector of quantities of goods which can be saved while keeping each agent's welfare unaffected. We characterized the family of *metrics* satisfying some intuitive properties (Theorem 1). This method of measuring gains is applicable to a wide variety of settings. It can be interpreted as generalizations of existing measures of welfare changes in single agent settings to multi-agent settings.

Then, we proposed an application to fair allocation. Based on Shapley's algorithm, we obtained a way of measuring each agent's contribution to the gains from trade. We declared an allocation fair if each agent receives her contribution to the gains from trade. We defined a fair allocation rule that assigns to each agent her contribution. This rule is inefficient, but we show that a recursive procedure, which is fair at each step of the recursion, yields an efficient rule (Theorem 2).

Now, we discuss relaxing some of the assumptions. First, we discuss relaxing the assumptions of boundary aversion and strict monotonicity of preferences. Then, we discuss relaxing some of the properties on *metrics*.

Throughout the paper, we assumed that preferences are strictly monotonic and satisfy boundary aversion. When preferences fail either of these properties but are (weakly) monotonic, Proposition 1 no longer holds. The *sets of possible gains from trade* of some economies are not strictly comprehensive; but they are still closed, convex, bounded, and comprehensive. Proposition 2 still holds. Moreover, for each closed, convex, bounded, and comprehensive set we can find an economy whose set of possible gains from trade and this set coincide.

The domain of closed, convex, bounded, and comprehensive sets is the usual domain of problems in bargaining theory. It is well-known that on this domain there is no *maximal* and *monotonic* solution. We can weaken *monotonicity* to hold whenever the smaller of the two sets of gains from trade is strictly comprehensive, and obtain a generalized version of the *weighted-gains* family. A member of this generalized family measures gains from trade by the largest vector proportional to a vector of weights,

[25]A similar result for the model of adjudicating conflicting claims holds (Dominguez 2006).

but, if this vector is not maximal, it drops some goods, and continues measuring gains from trade proportional to a restricted vector of weights. We refer to Thomson (2004) for a detailed treatment of this family in the context of bargaining theory.

For the application to fair allocation, *monotonicity* of the *metric* was necessary for the proof of Theorem 2. As stated in the text, we conjecture that an alternative proof can be obtained without *monotonicity* if we require a welfare improving property.

Finally, we discuss relaxing the requirement that *metrics* measure equal gains from trade in economies with equal *sets of possible gains from trade*. Sets of possible gains from trade depend on relatively little information about preferences. This property may be desirable when obtaining information is costly, but we may lose too much information in the aggregation procedure. Relaxing this property is an interesting an open question left for future research. For now, we note that *monotonicity* of a *metric* implies this property.

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