BAYESIAN ESTIMATION OF OUTSTANDING CLAIMS RESERVES

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Abstract

When a given process is seasonal and the seasonal pattern is stable the proportion of events that occur in a given fraction of the total period of the seasonality is constant over time. In this paper we present an application of Bayesian forecasting methods with stable seasonal patterns to insurance claims; specifically to claims incurred but not yet reported, IBNR. We assume that the time it takes for the claims to be completely paid is fixed and known, that payments are made annually and that the development of partial payments follow a stable pay-off pattern.

We present a Bayesian approach to forecasting either the total number claims or the total amount, given complete information for one or two past years and partial information on some development years for several occurrence years. Essentially the data corresponds to a run-off triangle in IBNR. Bayesian analysis of these problems allows the computation of forecasts for outstanding claims when there is not enough data to carry out time series analysis. Two different exact models are presented: one for discrete processes and one for continuous, which can be applied to forecast the number and amount of claims, respectively.

The advantage of using the Bayesian procedure presented here is that we can derive the complete predictive distribution of the reserve requirements, from which, in turn, we can obtain a point estimate as well as probability intervals. So the possible range of values can be formally quantified. We compare our method with others, using previously published data.

KEY WORDS: Chain-ladder technique, claims reserving, IBNR, Bayesian, predictive distribution.

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1. INTRODUCTION

Many existing models for claims reserving in non-life insurance assume, explicitly or implicitly, that the proportion of claim payments, payable in the *j-th* development period, is the same for all periods of origin, Taylor (1986). This situation is similar to forecasting with stable seasonal patterns. When a given process is seasonal and the seasonal pattern is stable the proportion of events that occur in a given fraction of the period of the seasonality is constant over time. Thus if the number of patients attended in a hospital is seasonal with a period of one year it would be expected that the number of patients attended each January is a fixed proportion of the annual figure; similarly for other months. If inflation (the rate of growth of prices) is seasonal then the proportion of the growth in prices, for a given month with respect to the total growth over the year, should be stable from year to year. These are two examples of processes with stable seasonal patterns: the first one discrete and the second continuous. Under these circumstances it is possible to use information on the seasonal pattern together with previous information to forecast the end-of-season total.

In de Alba and Mendoza (1995) we presented Bayesian models for forecasting with stable seasonal patterns when the data are integers. Several different applications are mentioned there and relevant references are also given. For an extensive discussion of situations where this kind of problems arise see Oliver (1987) and references therein. He presents a (multiplicative) forecasting model for cumulative events that, conditional on end-of-season totals being given and seasonal shape known, follow a multinomial distribution during the season. He then obtains the posterior distribution for end-of-season total.

In this paper we present an application of Bayesian forecasting methods with stable seasonal patterns to insurance claims. Specifically to the estimation of reserves for claims incurred but not yet reported. The basic difference is that instead of seasons we are dealing with development periods. We assume that the time it takes for the claims to be completely paid is fixed and known, that payments are made annually and that the development of partial payments follow a stable pay-off pattern. The proportion of claim payments, payable in the *j-th* development period, assumed to be the same for all periods of origin. This assumption is pointed out in the Introduction of Hesselager and Witting (1988). The results are thus applicable to any frequency of claim payments and length of pay-off' period. We present a Bayesian approach to forecasting total claims (number or amount) for the variable in question given a few (one or two) years of past data and information on some development years for several occurrence years. Essentially the data would correspond to a run-off triangle in IBNR. . We follow the terminology of Goovaerts et. al. (1990, p.243) in using the term IBNR reserves both “for claims as yet unreported, and a reserve for claims known to the company but not completely paid”. Our formulation of the problem is similar to Norberg (1986, 1993). Hesselager and Witting (1988) have extended his results using a credibility model. We do not use this latter approach here. We extend previous results using a specific full

Bayesian approach. Bayesian analysis of these problems allows the computation of forecasts for outstanding claims when there is not enough data to carry out time series analysis. Two different exact models are presented: one to forecast the number of outstanding claims and one for total claim amounts.

The paper is structured as follows. Section 2 gives a brief description of IBNR concepts. Section 3 describes the Bayesian model for discrete random variables that can be used to forecast the number of claims in IBNR. Section 4 presents a first continuous Bayesian model for claim amounts. Some examples are given in Section 5. Finally Section 6 presents a broader model for claim amounts, that we call a 'global' model, and Section 7 gives Final Remarks. All types of model are presented only in discrete time.

2. IBNR

The settlement of claims is usually subject to delay, and it is necessary for the insurer to set up reserve provisions for claims corresponding to losses which have been incurred by the insured during the covered period but which have not yet been settled. It is very important to estimate outstanding claims as accurately as possible in order to have a correct view of an insurer's financial situation. Methods used to estimate the necessary reserve provisions are usually classified as mechanical or non-stochastic and statistical or stochastic, Hossack et. al. (1983). Some of the mechanical methods can also be given a stochastic interpretation. Different statistical approaches for dealing with the problem have been developed in recent years. These methods attempt to find a consistent claim run-off pattern which has applied in the past; assume that pattern is stable and that it will continue to hold, and then apply that pattern to estimate the claims that have been incurred but are still outstanding. Adjustments may be made for anticipated future claim inflation, if necessary.

However, mechanical application of any statistical method does not lead to a 'correct' result, and the result obtained will often need to be heavily qualified. There may be trends in the settlement pattern, for example, that are difficult or impossible to explain, or there may be some doubt about the likelihood of an existing trend continuing in the future. In such circumstances, a great deal of reliance cannot be placed on the projected future run-off. Or, the settlement patterns of past years may be so erratic that, although a result can be obtained by mechanically applying one of the available methods, confidence in the final result will not be very great.

Even if the past settlement pattern is reasonably stable, the future run-off may be quite uncertain because of doubts that the pattern will continue, and because of claim inflation. The provision to be held will also be affected by assumed investment earnings. In practice, it is seldom possible to do any better than suggest a fairly wide range of not-unreasonable provisions based on different assumptions as to future claim inflation, investment earnings, etc., and possibly on different statistical methodology. For an extensive description of IBNR concepts and methods see Goovaerts et. al. (1990)

or Hossack et. al. (1983). The advantage of using the Bayesian procedure presented here is that we can derive the complete predictive distribution of the reserve requirements, from which, in turn, we can obtain a point estimate as well as probability intervals. So the possible range of values can be formally quantified. In a recent paper Doray (1995) presents a UMVUE of IBNR reserve in a lognormal regression model, in which he can compute its variance. In the last section of this paper we use his data and compare our results with his.

All IBNR methods, such as mechanical smoothing, least squares, credibility and regression can be applied to different claim figures, each of them leading to a specific forecasting method. Goovaerts et. al. (1990) describe some real and practical examples of insurance problems that can be solved by IBNR-techniques. In addition, they illustrate that not only loss reserving problems, but also any problem where a constant development pattern arises can be solved by these techniques, de Alba and Mendoza (1995) or Oliver (1987). The broad range of potential applications will be apparent from these references and those contained therein.

Bayesian analysis of IBNR reserves has been analyzed by Jewell (1989,1990), Verrall (1990) and Haastrup and Arjas (1996). Jewell (1989) formulated the problem considering that an IBNR "…claim in insurance is an event whose occurrence in some fixed exposure interval is not known until some later date because of random reporting delays." He considers that the natural formulation of the IBNR problem is in continuous time because of the underlying Poisson generation of claims and continuous reporting of delays. He goes on to formulate a continuous-time Bayesian model for predicting the total number of claims, assuming that the distribution of delays is uncertain. He chooses to analyze in detail only a single exposure year of unreported events. His model considers different types of data that may become available, such as occurrence and reporting dates. And concludes that at least the observation of reporting dates must be available. He uses the assumption of a homogeneous process and recognizes the fact that this may be unnecessarily restrictive. In a sequel paper Jewell (1990) considers the more realistic situation where reporting occurs only periodically, so that the reporting of dates is discretized into intervals of the basic exposure interval.

Verrall (1990) approaches the subject of predicting outstanding claims using hierarchical Bayesian linear models, considering the fact that the chain-ladder technique is based on a linear model; the two-way analysis of variance model (ANOVA). He essentially carries out a Bayesian analysis of the two-way ANOVA model to obtain Bayes and empirical Bayes estimates. The latter are given a credibility interpretation. In a more recent paper, Haastrup and Arjas (1996) model occurrences and development of claims as marked point process. A claim is described by an occurrence time, two covariates and a development. The latter contains a reporting delay, a settlement delay and a partial payment process. The distribution of the claims process is described by fourteen components. A priori, the intensities are assumed piecewise constant and their levels, as well as the intervals, are the parameters. This allows them to apply the nonparametric approach of Ferguson (1973). The analysis is done using Gibbs sampling. Verrall (1991) considers estimating claim reserves and outstanding claims by means of loglinear models. He compares these with maximum likelihood estimates. Renshaw and Verrall (1994) carry out the analysis assuming a Poisson model both for number of claims and claim amounts, while indicating that if a quasi-likelihood approach is taken then its use for data that are not positive integers will be justified. In Verrall (2000) he further explores the relation between the chain-ladder technique and some stochastic model. Again he assumes a Poisson distribution for claim amounts and indicates that this model should not necessarily be used for all data. He also points out that it is not clear how to deal with negative incremental claims.

We will use standard notation so that X_{it} = number (or amount) of events (claims) in *t-th* development year corresponding to year of origin (or accident year) *i*. Thus $\{X_{it}; i = 1,...,s\}$ where *s* = maximum number of years (sub periods) it takes to completely pay out the total number (or amount) of claims corresponding to a given exposure year. These random variables can be arranged in a matrix as in Table 1 below.

Table 1

year of origin	development year 1	2	...	t	...	s
1	X_{11}	X_{12}	...	X_{1t}	...	X_{1s}
2	X_{21}	X_{22}	...	X_{2t}	...	X_{2s}
3	X_{31}	X_{32}	...	X_{3t}	...	X_{3s}
:						
:						
k	X_{k1}	X_{k2}	...	X_{kt}	…	X_{ks}

For most IBNR methods it is usually assumed that *s=k* and the matrix is split into a set of known or observed variables (the upper left hand part) and a set of variables whose values are to be predicted (the lower right hand side). Thus we know the values X_{it} *for* $i + t \leq k + 1$, and the triangle of known values is the run-off triangle. This formulation agrees closely with Norberg (1986). It is the one we will use. We now describe the Bayesian procedures applicable to the estimation of claims reserves.

3. A MODEL FOR THE NUMBER OF CLAIMS

In de Alba (1988) and de Alba and Mendoza (1995) we consider a discrete process that can be stated in terms of IBNR as follows: let X_{it} = number of claims in *t-th* development year corresponding to year of origin (or accident year) *i*, i.e. $\{X_{it}; \ i = 1,...,k \ t=1,...,k\}$. Let *m*=number of complete years of origin for which there is complete data. Under the usual structure of the run-off triangle *m=1*, i.e. the only year of origin for which all the claims have been made is the first one. However, with our procedure we can easily accommodate any other value of m; in what follows we take $m=1$, so the available information is

$$\left\{X_{it};\quad i=1,\dots,k \quad t=1,\dots,k \quad i+t\le k+1\right\}.$$

Thus we have the run-off triangle:

Table 2

year of origin	development year 1	2	......	t	...	k-1	k
1	X_{11}	X_{12}	...	X_{1t}		$X_{1,k-1}$	X_{1k}
2	X_{21}	X_{22}	...	X_{2t}		$X_{2,k-1}$	-
3	X_{31}	X_{32}	...	X_{3t}		-	
:						-	-
k-1	$X_{k-1,1}$	$X_{k-1,2}$				-	-
k	X_{k1}	-					-

Let N_1=Total number of claims corresponding to year of origin 1, so that $\sum_{t=1}^{k} X_{1t} = N_1$. Assume that the number of claims in the *t-th* development year follows a Poisson distribution with parameter λ_t. This is similar to Assumption I in Norberg (1986), with fixed general risk conditions. However, we use a different approach based on conditional probabilities. Suppose N_1 is given, so that by conditioning on N_1 and assuming that the number of claims in one development year are independent of the number in another year, the vector $\underline{X}_1=(X_{11},X_{12},\dots,X_{1k})'$ has a multinomial distribution

$$\underline{X}_1 \big| N_1 = n_1 \sim MN(n_1, p_1,\dots,p_k), \tag{1}$$

with

$$p_t = \lambda_t \,/\, \sum_{t=1}^{k}\lambda_t ,$$

(Karlin (1968); page 185). Thus by conditioning on the total number of claims of origin year 1 we go from a parameterization in terms of λ's to one in terms of the p's. From the assumption of a stable pattern of development, which is also included in Norberg (1986), it follows that p_t does not depend on the year of origin i. If $m>1$ we could let $\underline{X}^*=\underline{X}_1+\dots+\underline{X}_m$ so that

$$\underline{X}^* \big| N_1 = n_1,\dots,N_m = n_m \sim MN(n_1+\dots+n_m,\quad p_1,\dots,p_k). \tag{2}$$

In the IBNR setup we are interested in estimating $N_2,N_3,N_4,\dots,N_k$, the total claims for each year of origin, using the available information in the run-off triangle. We will obtain an estimate of N_i, $i=2,...,k$. We first describe our method for forecasting the number of outstanding claims in year of origin 2, i.e. we first derive the results for N_2 and later extend them for $i=3,...,k$. From the multinomial distribution in (1) it follows that for some $a \le k$ and $X_i^* = \sum_{t=1}^{a} X_{it}$

$$X_i^* \big| N_i \sim BI(n_i, p_a^*) \qquad p_a^* = p_1 + \ldots + p_a \qquad i = 2,3,\ldots,k\,. \tag{3}$$

Specifically for $i=2$ we are interested in the case $a=k-1$. Since the number of claims in each development year for every year of origin are independent, which follows directly from the assumption of Poisson distribution of claims mentioned above, then from equation (3) the joint density of X_i^* , $i = 1,2,$ is

$$f_{x_1^* x_2^*}(x_1^*, x_2^* | n_1, n_2, p_a^*) = \binom{n_1}{x_1^*}\binom{n_2}{x_2^*}(p_a^*)^{x_1^*+x_2^*}(1-p_a^*)^{n_1+n_2-x_1^*-x_2^*}\,.$$

Given the observations on x_1^*, x_2^* and n_1 (denote this by D_{1a}) this is the Likelihood function for n_2 and p_a^*. Now we take a diffuse non-informative joint prior density for n_2 and p_a^* assuming independence. Jeffreys' prior for p_a^* is

$$f(p_a^*) \;\propto\; (p_a^*)^{-1/2}(1-p_a^*)^{-1/2}\,. \tag{4}$$

Considering that $N_2>0$ we use, de Alba and Mendoza (1995), the prior

$$f(n_2) \;\propto\; 1/n_2\,. \tag{5}$$

Combining the priors and the Likelihood we obtain the marginal posterior distribution for p_a^* as:

$$f(p_a^* | D_{1a}) \;=\; \binom{n_1}{r}(p_a^*)^{r_2-1/2}(1-p_a^*)^{n_1-r_2-1/2}, \quad 0 < p_k^* < 1, \tag{6}$$

with $r_2 = \sum_{t=1}^{a} x_{1t}$. This is a Beta$(\alpha_2,\beta_2)$, where $\alpha_2=r_2+1/2$ and $\beta_2=n_1-r_2+1/2$. The marginal posterior for N_2 is

$$f_{N_2}(n_2 | D_{1a}) = \frac{B(\alpha_2 + x_2^*, \beta_2 + n_2 - x_2^*)}{B(\alpha_2, \beta_2)}\binom{n_2-1}{n_2-x_2^*}, \tag{7}$$

a Beta-Pascal distribution, Raiffa and Schlaifer (1961), with $n_2 \ge x_2^*$. Hence the posterior mean is

$$E(N_2 | D_{1a}) = x_2^*(\alpha_2 + \beta_2 - 1)/(\alpha_2 - 1) \qquad (8)$$

and variance

$$Var(N_2 | D_{1a}) = x_2^*(x_2^* + \alpha_2 - 1)(\alpha_2 + \beta_2 - 1)\beta_2 / (\alpha_2 - 1)^2(\alpha_2 - 2). \qquad (9)$$

The posterior mean is the point estimator for N_2 provided we assume a quadratic loss function. This constitutes an estimator for the total number of claims corresponding to accident year 2 and development year *k*. It can be used to determine the unobserved claims by subtracting known claims from this number. Let R_2 be the number of unobserved claims for accident year 2, then the Bayesian estimator of R_2 is

$$E(R_2 | D_{1a}) = E(N_2 | D_{1a}) - \sum_{t=1}^{k-1} X_{2t}$$

and $Var(R_2 | D_{1a}) = Var(N_2 | D_{1a})$.

To estimate the unknown number of claims corresponding to accident year 3 we proceed as follows. Consider the portion of the runoff triangle given in Table 3 below.

Table 3

year of origin	development year 1	2	...	k-2	k-1	k
1	X_{11}	X_{12}	...	$X_{1,k-2}$	$X_{1,k-1}$	$X_{1,k}$
			...			
3	X_{31}	X_{32}	...	$X_{3,k-2}$	-	-

We use the same formulas as above but now for *i=3*, with $D_{2a} = \{x_1^*, x_3^*, n_1\}$. We are interested in the case $a = k - i + 1 = k - 2$, $X_3^* = \sum_{t=1}^{a} X_{3t}$ and $r_3 = \sum_{t=1}^{a} x_{1t}$. Hence the posterior mean of the unobserved number of claims is

$$E(R_3 | D_{1a}, D_{2a}) = x_3^*(\alpha_3 + \beta_3 - 1)/(\alpha_3 - 1) - \sum_{t=1}^{k-2} X_{3t} \qquad (10)$$

and the variance

$$Var(R_3 | D_{1a}, D_{2a}) = x_3^*(x_3^* + \alpha_3 - 1)(\alpha_3 + \beta_3 - 1)\beta_3 / (\alpha_3 - 1)^2(\alpha_3 - 2), \qquad (11)$$

where now $\alpha_3 = r_3 + 1/2$ and $\beta_3 = n_1 - r_3 + 1/2$. The process can be continued for

R_4 to R_k in a similar fashion. We assume that the total number of claims corresponding to one accident year is independent of the total number corresponding to any other accident year. This is the same as Assumption IV in Norberg (1986). Then the Bayesian estimator of the total number of claims for a specific insurance business is

$$E(R_T | D) = \sum_{i=2}^{k} E(R_i | D) \tag{12}$$

with variance

$$Var(R_T | D) = \sum_{i=2}^{k} Var(R_i | D) . \tag{13}$$

In the expressions above $D = \{ x_i^*, i = 1,\ldots k, n_1 \}$.

The main purpose in IBNR is to compute the required reserves. With our method this can be done without actually estimating the required payments in each one of the individual future development years for each accident year. Nevertheless it is also possible to estimate each one separately. The formulas are essentially the same. To illustrate consider the first three rows of Table 2, which are reproduced below. When estimating total reserves for each i (row), we computed $\sum \boldsymbol{F}_{it}$ where the sum is over values of the subscripts such that $t \le k - i + 1$. That is, the lower right triangle. In the third row we estimated $\boldsymbol{F}_3 = \boldsymbol{F}_{3,k-1} + \boldsymbol{F}_{3,k}$. To estimate $\boldsymbol{F}_{3,k-1}$ individually we use the previous formulas with only *k-1* development years for which we have complete data (up to *k-1*) for two accident years, i.e. *m=2*, de Alba and Mendoza (1995). Having estimated $\boldsymbol{F}_{3,k-1}$ we can obtain $\boldsymbol{F}_{3,k}$. If necessary this process can be repeated cell by cell.

year of				development year			
origin	1	2	......	t	...	k-1	k
1	$\boldsymbol{X_{11}}$	$\boldsymbol{X_{12}}$	...	$\boldsymbol{X_{1t}}$		$\boldsymbol{X_{1,k-1}}$	$\boldsymbol{X_{1k}}$
2	$\boldsymbol{X_{21}}$	$\boldsymbol{X_{22}}$	...	$\boldsymbol{X_{2t}}$		$\boldsymbol{X_{2,k-1}}$	$\boldsymbol{F_{2k}}$
3	$\boldsymbol{X_{31}}$	$\boldsymbol{X_{32}}$	...	$\boldsymbol{X_{3t}}$		$\boldsymbol{F_{3,k-1}}$	$\boldsymbol{F_{3,k}}$

The results given in equations (12) and (13) assume that in deriving the posterior distribution of the reserves for a given accident year the data used is from the *m* complete accident years and the one being estimated. Thus if we are estimating reserves for the *j-th* accident year we are leaving out the information available for years *i=1,...,j-1*. The posterior mean can easily be modified to include this information. To do this make the following definitions. In general at *i-th* accident year let $a_i = k-i+1$, $i=2,\ldots,k$, $X_i^* = \sum_{t=1}^{a_i} X_{it}$ and $r_i^* = \sum_{j=1}^{i-1} \sum_{t=1}^{a_i} X_{jt}$, $n_i^* = \sum_{t=1}^{k} X_{it}$. It can be shown that

$$E(N_i \middle| D_{ja_j}; \quad j=1,\ldots,i-1) = X_i^* \left(\frac{\sum_{j=1}^{i-1} E(n_i^*)}{r_i^* + 1/2} \right), \quad j=2,\ldots,k .$$

The posterior variance can not be obtained analytically. However an approximation can be obtained by simulation as follows. For $i=2,...,k$ generate an observation for $p_{a_i}^*$ using (6) with $\alpha_i^* = r_i^* + 1/2$ and $\beta_i^* = \sum_{i=i}^{i-1} n_i^* - r_i^* + 1/2$ and then use this value to generate an observation from $f(n_2 | p_a^*, D_{1a_1})$ which is $NB(X_2^*, p_a^*)$. The process is repeated for $n_3^*,\ldots,n_k^*$. Wherever needed the generated observations are used in the fórmulas for succeeding values. For example, to generate an observation for n_3^* we need n_1^* and n_2^*, but the latter is unknown so we use the simulated value, thus taking into account all the variability. In turn generated valued for n_2^* and n_3^* are utilized to generate n_4^*, and so on

4. A 'YEAR BY YEAR' MODEL FOR AGGREGATE CLAIMS

In this section we present a model for continuous random variables which can be used to estimate the total of unobserved claim amounts and hence the necessary reserves for outstanding claims. Let the random variable $Z_{it}^* \geq 0$ represent the value of aggregate claims the *t-th* development year of accident year *i* , *i=1,...,k*, and *t=1,...,s*. As in the discrete case let *m*=number of years of origin, or business years, for which there is complete data. As above, in this paper we refer to years of origin and development years, but the procedures are equally applicable with other time units. Under the usual structure of the run-off triangle *m=1*, i.e. the only year of origin for which the total claim amount has been paid is the first one. However, with our procedure we can easily accommodate any other value of *m*; in what follows we take *m=1* and use *k=s*, so the Z_{it}^* are known for $i+t \leq k+1$. We assume

$$Y_{it} = \log\left(\frac{\mathrm{Z}_{\mathrm{it}}^*}{\mathrm{X}_{\mathrm{it}}}\right) = \log(Z_{it}), \qquad (14)$$

where we can use alternative specifications for X_{it}. The first one we consider is to let , as before, X_{it} be the number of closed claims in *t-th* development year corresponding to year of origin *i*. Hence, in this case Z_{it} is the corresponding average claim. This is the structure used in Taylor and Ashe (1983). The second specification we consider in this paper is that $X_{it} = X_i =$ some measure of exposures in each year of origin. It is used as a standardizing measure of business volume. This is the formulation used in most of the references, e.g. Verrall (1990), Renshaw and Verrall (1994), England and Verrall

(1999) and Doray (1996). Since the first formulation is more general we shall consider it first.

In either case, we assume in addition that

$$Y_{it} = \mu + \alpha_i + \beta_t + \varepsilon_{ij} \qquad \varepsilon_{ij} \sim N(0, \sigma^2)$$

$i=1,...,k$, $t=1,...,k$ and $i+t \leq k+1$ i.e. an unbalanced two-way analysis of variance model, Verral (1990) and Doray (1996) and England and Verrall (1999). Thus Z_{it} follows a log-normal distribution,

$$f(y_{it} \mid \mu, \alpha_i, \beta_t, \sigma^2) \propto \frac{1}{\sigma} e^{-\frac{1}{2\sigma^2}(y_{it} - \mu - \alpha_i - \beta_t)^2} . \qquad (15)$$

4.1 LOSS RESERVING USING CLAIMS PER PERIOD

We now want to estimate (or obtain the distribution of) aggregate claims for accident year *i* given information on at least one development year and perhaps on *m* previous completely known accident years. Hence, using the run-off triangle setup, we are really interested in estimating $Z^*_{i0} = \sum_{t=1}^{k} Z^*_{it}$, $i=2,...,k$, given Z^*_{10}, X_{it} and Z^*_{it}, $i=1,...,k$ $t=1,...,k$, with $i+t \leq k+1$. Now let $R_i = \sum_{t=a+1}^{k} Z^*_{it} = Z^*_{i0} - Z^*_{ia_i}$, for $i=2,...,k$ with $Z^*_{ia_i} = \sum_{t=1}^{a_i} Z^*_{it}$ $a_i=k-i+1$, i.e. R_i = the total of the aggregate claims process for the development years for which it is unknown, and corresponding to business year *i*.

We use the 'profile' likelihood with σ^2 replaced by the mode of its posterior distribution under a non-informative prior,

$$\tilde{\sigma}^2 = \frac{\nu s^2}{\nu - 1}, \qquad \nu = T_U\text{-}2k+1$$

Verral (2000), and carry out the rest of the analysis with this value taken as given and replaced in (15). Hence, using matrix notation

$$f(\underline{y} \mid \underline{\theta}, \tilde{\sigma}^2, X, \underline{x}_1, \underline{x}_2, ..., \underline{x}_k) \propto \tilde{\sigma}^{-T_u} e^{-\frac{1}{2\tilde{\sigma}^2}(\underline{y} - X\underline{\theta})'(\underline{y} - X\underline{\theta})} ,$$

where $\underline{y} = \{y_{it}$, $i=1,...,k$, $t=1,...,k$ and $i+t \leq k+1\}$, $\underline{\theta}$ is a vector of parameters and X is the design matrix in (15). The vectors $\underline{x}_1 = (x_{11}\ x_{12} .. \ x_{1k})'$, $\underline{x}_2 = (x_{21}\ x_{22} .. \ x_{2k-1})'$, …, $\underline{x}_k = (x_{k1})$ contain the known data in each row of the triangle. Also, we define T_U = number of cells with known claim information in the upper triangle and T_L = number of cells in the lower triangle, whose claims are unknown. It is well known in ANOVA that certain restrictions must be imposed on the parameters in order to attain estimability. One option is to use $\sum_{i=1}^{n}\alpha_i = \sum_{j=1}^{n}\beta_j = 0$, Kremer (1982). We use the alternative assumption that $\alpha_1 = \beta_1 = 0$ so then $\underline{\theta}' = (\mu\ \alpha_2 \ldots \alpha_k\ \beta_2 \ldots \beta_k)$.

Using the same assumptions about the distribution of the number of claims and notation as in Section 2, we can rewrite expression (1) as

$$f(\underline{x}_1 \mid n_1, \underline{p}) = \frac{n_1!}{\prod_{t=1}^{k} x_{1t}!} p_1^{x_{11}} .. p_k^{x_{1k}} .$$

Using properties of the Multinomial distribution we can also write

$$f(\underline{x}_2 \mid n_2, \underline{p}) = f(x_2^* \mid n_2, \underline{p}_{k-1}^*) f(x_{21}..x_{2k-1} \mid x_2^*, p_1..p_{k-1})$$

$$f(\underline{x}_3 \mid n_3, \underline{p}) = f(x_3^* \mid n_3, p_{k-2}^*) f(x_{31}..x_{3k-2} \mid x_3^*, p_1..p_{k-2})$$

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$$f(\underline{x}_k \mid n_k, \underline{p}) = f(x_k^* \mid n_k, p_1^*) .$$

We assume independence between the average claim and the number of claims. The joint pdf for each cell is then

$$f^*(\underline{y}, \underline{x}_1, .. \underline{x}_k \mid \underline{\theta}, n_1, n_2 ... n_r, \underline{p}, \tilde{\sigma}, X) = f(\underline{y} \mid \underline{\theta}, \tilde{\sigma}, X) \times \prod_{t=1}^{k} f(\underline{x}_t \mid n_t, \underline{p}) \quad . \quad (16)$$

We integrate (16) over the variables $(x_{21}\, x_{22} \ldots x_{2k-1} \ldots x_{31}\, x_{32} \ldots x_{3k-2} \ldots x_{k-11}\, x_{k-12})$ that do not contain 'usable' information on N_i nor on $\underline{p}$ and use the resulting 'integrated' likelihood

$$f(\underline{y}, \underline{x}_1, .. \underline{x}_k \mid \underline{\theta}, n_1, n_2 ... n_r, \underline{p}, \tilde{\sigma}, X) = f(\underline{y} \mid \underline{\theta}, \tilde{\sigma}, X) \times \prod_{i=1}^{k} f(x_i^* \mid n_i, p_{a_i}^*) , \quad (17)$$

where $a_i = k - i + 1$ and recalling $x_i^* = \sum_{t=1}^{a_i} x_{it}$. This will alow us to obtain separate estimators of the claim reserves for each year of origin. We see that (17) includes the following elements:

1) $$f(x_i^* \mid n_i, p_{a_i}^*) = \frac{n_i!}{x_i^*!(n_i - x_i^*)!}(p_{a_i}^*)^{x_i^*}(1 - p_{a_i}^*)^{n_i - x_i^*}, \quad (18)$$

i = 2,…,k, which contains the useful information about N_i.

2) $$f(x_1^* \mid n_1, p_1, p_2, ..., p_k) = \frac{n_1!}{x_1^*!\, x_{1a_{i+1}}!..x_{11}!} p_{a_i}^{*x_1^*} p_{a_{i+1}}^{x_{1a_{i+1}}} ..p_1^{x_{11}}, \quad (19a)$$

or

$$f(x_1^* \mid n_1, p_1, p_2, ..., p_k) = \frac{n_1!}{\prod_{t=1}^{k} x_{1t}!} p_1^{x_{11}} ..p_k^{x_{1k}} \quad (19b)$$

which contains relevant information about $\underline{p}$. The only complete information about the p's is available in the years of origin for which the claims have been fully paid. In this paper we are assuming there is only one, but this can easily be modified. Finally, the last component of the likelihood, that contains information about $\underline{\theta}$, is

3) $$f(\underline{y} \mid \underline{\theta}, \tilde{\sigma}) \propto \tilde{\sigma}^{-T_U} e^{-\frac{1}{2\tilde{\sigma}^2}(\underline{y} - X\underline{\theta})'(\underline{y} - X\underline{\theta})}. \quad (20)$$

Next we specify a prior distribution for the parameters. Assume they are independent a-priori so that

$$f(\underline{\theta}, n_2, ..., n_k, \underline{p}) = f(\underline{\theta}) \prod_{i=2}^{k} f(n_i) f(\underline{p})$$

and with non-informative priors

$$f(\underline{\theta}, n_2, ..., n_k, \underline{p}) \propto \prod_{i=2}^{k} (1/n_i),$$

Zellner (1971), equation (5) above and O'Hagan (1994) for $\underline{\theta}$, n_i and $\underline{p}$ respectively. The prior for $\underline{p}$ is a *Dirichlet(1,1,…,1)*. The joint posterior distribution is then

$$f(\underline{\theta}, n_2, ..., n_k, \underline{p} | D) \propto e^{-\frac{1}{2\tilde{\sigma}^2}(\underline{y} - X\underline{\theta})'(\underline{y} - X\underline{\theta})} \frac{n_1!}{x_1^*! x_{1a_{i+1}}! .. x_{11}!} p_{a_i}^{*x_1^*} p_{a_{i+1}}^{x_{1a_{i+1}}} ... p_1^{x_{11}} \times$$

$$\prod_{i=2}^{k} \frac{(n_i - 1)!}{(x_i^* - 1)!(n_i - x_i^*)!} (p_{a_i}^*)^{x_i^*} (1 - p_{a_i}^*)^{n_i - x_i^*}, \qquad (21)$$

where D represents all the known information included in the posterior distribution, i.e. $D = \{x_{11}, .. x_{1k}, x_1^*, x_2^*, ..., x_k^*, n_1, X, \underline{y}\}$. We can rewrite (21) as

$$f(\underline{\theta}, n_2, ..., n_k, \underline{p} | D) \propto f(\underline{\theta} | D) \times \prod_{i=2}^{k} f(n_i | \underline{p}, D) \times f(\underline{p} | D),$$

where

$$f(\underline{\theta} | D) \propto e^{-\frac{1}{2\tilde{\sigma}^2}(\underline{\theta} - \hat{\underline{\theta}})' X' X (\underline{\theta} - \hat{\underline{\theta}})}, \qquad (22)$$

with $\hat{\underline{\theta}} = (X' X)^{-1} X' \underline{y}$. Furthermore

$$f(n_i | \underline{p}, D) = \frac{(n_i - 1)!}{(x_i^* - 1)!(n_i - x_i^*)!} (p_{a_i}^*)^{x_i^*} (1 - p_{a_i}^*)^{n_i - x_i^*}, \qquad (23)$$

$i = 1,2,\ldots,k$, and

$$f(\underline{p} | D) = \frac{n_1!}{\prod_{t=1}^{k} x_{1t}!} p_1^{x_{11}} .. p_k^{x_{1k}}. \qquad (24)$$

To compute the reserves for the total outstanding claims we need to estimate the lower portion of the triangle. We do this by obtaining the mean and variance of the predictive distribution. Hence for each cell we have:

$$E(Z_{it}^* | D) = E(Z_{it} x_{it} | D) = E(Z_{it} | D) E(x_{it} | D)$$

by the independence of $\underline{Z}$ *and* $(\underline{x}_1, .. \underline{x}_k)$. Now

$$E(x_{it} | D) = \underset{n_i, \underline{p}}{E}\left[E(x_{it} | D, n_i, \underline{p})\right] = \underset{\underline{p}}{E}\left\{\underset{n|\underline{p}}{E}\left[E(x_{it} | n_i, \underline{p}) | p, D\right] | D\right\}$$

and $E(x_{it} | n_i, \underline{p}) = n_{it} = N_i p_t$. So that

$$\underset{n|\underline{p}}{E}\left[E(x_{it} \mid n_i, \underline{p}) \middle| p, D\right] = \underset{n|\underline{p}}{E}\left[n_i p_t \mid \underline{p}, D\right] = p_t \underset{n|\underline{p}}{E}\left[n_i \mid \underline{p}, D\right]$$

and, from equation (23),

$$\underset{n|\underline{p}}{E}\left[n_i \middle| \underline{p}, D\right] = \frac{x_i^*}{p_{a_i}^*}.$$

Hence

$$E(x_{it} \mid D) = x_i^* \underset{\underline{p}}{E}\left[\frac{p_t}{p_{a_i}^*} \middle| D\right] = \frac{x_i^*}{x_{1a_i}} \cdot x_{1t} = \left(\frac{x_i^*}{x_{1a_i}}\right) n_1 \cdot \left(\frac{x_{1t}}{n_1}\right).$$

In this last expression the first term in parenthesis is the proportional increase between the i-thyear and year 1, computed with the information available for both years. Then this increase is applied to n_1 by multiplying yielding an 'estimate' of n_2. The last term is an 'estimate' of the corresponding probability of t^{th} development year. Finally, putting all results together we have that

$$E(Z_{it}^* \mid D) = e^{\hat{\mu} + \hat{\alpha}_i + \hat{\beta}_t + \frac{1}{2}\tilde{\sigma}^2 v_{it}} \frac{x_i^*}{x_{1a_i}} \cdot x_{1t}, \qquad (25)$$

where $\tilde{\sigma}^2 v_{it}$ is the variance of the *(i,t)-th.* cell, for which a forecast is being made. Then the Bayes estimate of outstanding claims for year of business i is $\sum_{t>k-i+1} E(Z_{it}^* \mid D)$ and the Bayes 'estimate' of the variance (the predictive variance) is

$$\sum_{t>k-i+1}\left[Var(Z_{it}^* \mid D) + 2\sum_{s>t} Cov(Z_{is}^*, Z_{it}^* \mid D)\right]. \qquad (26)$$

Let the vector b_{it} be such that $E(Y_{it}) = b_{it}^{'}\underline{\hat{\theta}} = \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_t$, for the 'future' values of i and t, i.e. $t > k - i + 1$. EMBEDTherefore, the variance of Y_{it} is

$$Var(Y_{it}) = \tilde{\sigma}^2\left[1 + b_{it}^{'}(X'X)^{-1} b_{it}\right] = \tilde{\sigma}^2 v_{it}.$$

Since Z_{it} follows a log-normal distribution, its mode, median and expected values are then

mode $(Z_{it}) = e^{b_{it}^{'}\underline{\hat{\theta}} - \tilde{\sigma}^2[1 + b_{it}^{'}(X'X)^{-1} b_{it}]} = e^{b_{it}^{'}\underline{\hat{\theta}} - \tilde{\sigma}^2 v_{it}}$, median $(Z_{it}) = e^{b_{it}^{'}\underline{\hat{\theta}}}$, and

$E(Z_{it}) = e^{b_{it}^{'}\underline{\hat{\theta}} + \tilde{\sigma}^2[1 + b_{it}^{'}(X'X)^{-1} b_{it}]/2} = e^{b_{it}^{'}\underline{\hat{\theta}} + \tilde{\sigma}^2 v_{it}/2}$. The variance of Z_{it} is:

$$Var(Z_{it}) = e^{2b_{it}'\hat{\underline{\theta}} + \tilde{\sigma}^2 v_{it}}(e^{\tilde{\sigma}^2 v_{it}} - 1).$$

In order to compute equation (26) we need to find $Cov(Z_{is}^*, Z_{it}^*)$. But this involves both $\underline{Z}$ and $\underline{x}$. When the latter is not involved Doray (1996) does this based on the fact that the vector of 'future' values $\mathbf{y} = (Y_u)$ has a multivariate log-normal distribution. Then, following Crow and Shimizu (1988) he gets

$$Cov(Y_{it}, Y_{js}) = e^{b_{itb}'\hat{\underline{\theta}} + b_{js}'\hat{\underline{\theta}} + \tilde{\sigma}^2(v_{it} + v_{js})/2}(e^{\rho\tilde{\sigma}^2(v_{it}v_{js})^{1/2}} - 1),$$

where $\rho(Y_{it}, Y_{js}) = \rho$ is the correlation between Y_{it} and Y_{js}. Thus the covariance for each pair of elements in the lower triangle would need to be evaluated to find the variance of the IBNR claims.

It is clear that these formulas can be very cumbersome to compute, Doray (1996), England and Verrall (1999) and Verrall (1991). However, it is relatively easy to obtain the distribution of the reserves by simulation, as follows: for $j = 1,\ldots, N$

a) Generate a vector from the Dirichlet posterior: $\underline{p}^{(j)}$, equation (24),
b) Use $\underline{p}^{(j)}$ and (23) to generate an observation for each n_i. For each one we use the appropriate definition of $p_{a_i}^*$. Thus we generate $(n_2 n_3, .., n_k) = \underline{n}^{(j)}$.
c) Use $\underline{n}^{(j)}, \underline{p}^{(j)}$ to generate observations for the unknown portion of $\underline{x}_i^{(j)}$ from each of $(k - 1)$ multinominals (one for each year) $f(\underline{x}_i^{(j)} \mid n_i^{(j)}, \underline{p}^{(j)})$ as defined in equation (19b). The process up to here provides a set of randomly generated values for the number of claims in each cell of the (unobserved) lower right triangle.
d) Generate an observation from $N(\hat{\underline{\theta}}, \tilde{\sigma}^2(X'X)^{-1}) \rightarrow \underline{\theta}^{(j)}$. We do this by using the Choleski decomposition.
e) Compute $Y_{it}^{(j)} = \mu^{(j)} + \alpha_i^{(j)} + \beta_t^{(j)}$ each (i,t), $t > k - i + 1$ and $Z_{it}^{(j)} = exp\{Y_{it}^{(j)}\}$. The previous expression is equivalent to using the median of the predictive distribution of $Y_{it}^{(j)}$, which yields a more conservative estimate than the mean and less than the corresponding mode, Doray (1996). This part of the process (steps d) and e)) provide a set of randomly generated values for the median (average) claim in each cell of the (unobserved) lower right triangle.
f) Compute $Z_{it}^{*(j)} = x_{it}^{(j)} * Z_{it}^{(j)}$ for each $i = 2,..,k$, $t > k - i + 1$
g) To obtain a value of the total required reserves add all the $Z_{it}^{*(j)}$ i.e. $R^{(j)} = \sum_{i,t} Z_{it}^{*(j)}$.
h) The mean variance can be computed as

$$\sigma_R^2 = \frac{1}{N}\sum_{j=1}^{N}\frac{(R^{(j)} - \overline{R})^2}{N} \quad and \quad \overline{R} = \frac{1}{N}\sum_{j=1}^{N} R^{(j)},$$

and the histogram of the *N,* $R^{(j)}$'s can be used to analyze the behavior of reserves.

This procedure can be used in the presence of negative incremental claims, as follows. Notice that equation (24) depends only on the number of claims in each development year for the first business year. Hence, as long as the negative increment does not occur in one of these cells, then except for d) the steps in the simulation will not be affected. We assume that the average claim amount are only affected by the negative increment in the cell in which it appears and perhaps also in the one after it. Then we can use the remaining cells to obtain $(\underline{\hat{\theta}},\tilde{\sigma}^2)$ and carry out the remaining steps. This will be explored in detail in another paper although we do provide an example in Section 6.

4.2 LOSS RESERVING USING A MEASURE OF EXPOSURE PER ACCIDENT YEAR

In this Section we present the second specification for equation (14). Now in it let $X_{it} = X_i =$ some exposure factor or the (**known**) exposures in each year of origin, Verrall (1990), Renshaw and Verrall (1994), England and Verrall (1999). Since X_i, $i = 1,\ldots,k$, are no loger random variables we only need to model Z_{it} as before and then equation (25) simplifies to

$$E(Z_{it}^* \mid D) = e^{\hat{\mu}+\hat{\alpha}_i+\hat{\beta}_t+\frac{1}{2}\tilde{\sigma}^2 v_{it}} \cdot X_i. \qquad (27)$$

The expressions for $Var(Z_{it}^* \mid D)$ and $Cov(Z_{is}^*, Z_{it}^* \mid D)$ will simplify somewhat, but they will still be cumbersome to compute. Hence we obtain the distribution by simulation but we do not need steps a)-c) above. We start with step d) and step f) will be modified as follows

f) Compute $Z_{it}^{*(j)} = X_i * Z_{it}^{(j)}$, all $i = 2,..,k$, $t > k-i+1$.

Steps g) and h) remain unchanged. Using this procedure we can obtain the predictive distribution of the reserves that will be comparable to those given in the references, but with the advantage of having the complete predictive distribution.

5. A 'GLOBAL' MODEL FOR IBNR AGGREGATE CLAIM AMOUNTS

The above results assume that in deriving the predictive distribution of the reserves for a given accident year the data used are from the *m* complete accident years (usually *m = 1*) and the one being estimated. Thus if we are estimating reserves for the *j-th* accident year we are leaving out the information available for years *i=1,...,j-1*. The information that is left out is essentially on $\underline{p}$, since all the information on $\underline{\theta}$ is contained in the $\underline{Z}$'s and has been used in the estimation/simulation process. The analysis can easily be modified to include all the information on $\underline{p}$. This is done as follows: in the Appendix we show that

$$f(\underline{p}\,|\,D)$$

$$\propto f(p_k\,|\,D)\cdot f(p_{k-1}\,|\,p_k, D)\cdot f(p_{k-2}\,|\,p_{k-1}, p_k, D)\ldots f(p_2\,|\,p_3, p_4 \ldots p_k, D) f(p_1\,|\,p_2 .. p_k, D)$$

where

$$f(p_{k-i}\,|\,p_{k-i+1}, p_{k-i+2}, .. p_k, D) \propto \left(\frac{p_{k-i}}{p^*_{k-i}}\right)^{a_{k-i}-1}\left(1-\frac{p_{k-i}}{p^*_{k-i}}\right)^{b_{k-i}-1},$$

with $a_{k-i} = \sum_{j=1}^{i+1} x_{jk-i} + 1$ and $b_{k-i} = \sum_{t=1}^{k-i-1}\sum_{j=1}^{i+1} x_{jt} + 1$. We also show that the posterior mean of the outstanding claims in the *i-th*. accident year is given by:

$$E(OC_i\,|\,D) \cong \left[\sum^{k-(i-1)} x_{it} \Big/ \left(\sum_{t=1}^{k-(i-1)} (x_{1t} + x_{2t} + .. + x_{(i-1)t})\right)\right] \times \left[\sum_{j=1}^{i-1} \sum_{t=k-(i-2)}^{k*} x_{jt}\right],$$

where $x_{jt} = E(OC_j\,|\,D)$ whenever $j + t > k + 1$. This result is also very similar to that of Verrall (2000).

We can use the previous results to obtain the posterior distribution of the required reserves by Monte Carlo Simulation. We only need to replace step a) of the process given in Section 4 by the following steps:

a.1) Use expression B_1 (in the Appendix) to generate $p_k^{(j)}$ from a

$$\text{Beta}\left(x_{1k} + 1, \sum^{k-1} x_{1t} + 1\right)$$

a.2) Use expression B_2 (in the Appendix) to generate $\tilde{\theta}_{k-1}^{(j)}$ from a.3) Beta$\left(x_{1k-1} + x_{2k-1} + 1, \sum_{t=1}^{k-2} (x_{1t} + x_{2t}) + 1 \right)$

a.4) Use the results of 1) and 2) to generate $p_{k-1}^{(j)} = \tilde{\theta}_{k-1}^{(j)} (1 - p_k^{(j)})$

a.5) Use expression B_3 (in the Appendix) to generate $\tilde{\theta}_{k-2}^{(j)}$ from

Beta$\left(x_{1k-2} + x_{2k-2} + x_{3k-2} + 1, \sum_{t=1}^{k-3} (x_{1t} + x_{2t} + x_{3t}) + 1 \right)$

a.6) Use the results of 1) - 5) to generate $p_{k-2}^{(j)} = \tilde{\theta}_{k-2}^{(j)} (1 - p_{k-1}^{(j)} - p_k^{(j)})$,

and so on. With this we will have generated a vector $(p_k^{(j)}, p_{k-1}^{(j)}, \ldots, p_1^{(j)})$ which now can be used in the following steps of the simulation, as in Section 4 . Notice that since the global model is arrived at using all the information on $\underline{p}$ and this vector does not appear when using $X_{it} = X_i$ = exposures per busines or accident year, as in the 'exposure' model of Section 4.2, then there is no need to use a global model when X_{it} is defined this way.

6. APPLICATIONS

To exemplify the application of the methods presented in the previous sections we use two sets of data. First we use the data given in Tables 4.8 and 4.10 of Brown (1993). They are reproduced as Tables 4 and 5, below. Later we will also use the data of Verrall (1990), originally due to Taylor and Ashe (1983).

Table 4

Accident Year	Cumulative Loss Payments					Ultimate Number of Claims
	Development Year					
	0	1	2	3	4	
1988	2,000	6,000	9,000	11,200	14,000	1,000
1989	2,600	6,840	10,920	15,600		1,200
1990	2,380	8,960	14,400			1400
1991	3,120	10,800				1500
1992	3,800					1500

With the data from Brown (1993) we first analyze the number of claims using the discrete model and then aggregate claims using both of the continuous year-by-year models. The number of claims are given in Table 5. The estimated number of outstanding claims obtained by the Bayesian method using the predictive mean, as described in Section 3, is 1648. Figure 1 shows the corresponding distribution obtained with 5000 simulations.

The estimate of required reserves for IBNR aggregate claims is obtained first by the Bayesian method using an 'average' estimator per cell, it is \$49,481. With this approach the predictive mean of the number of claims is multiplied by the predictive mean (or median) of the average claim in each cell. It is the easiest to apply and the variance is not computed, since it requires estimating the variances and the second moments. It is included for the sake of comparing. The estimate with the straightforward classical chain-ladder model, without including neither exposures, nor closed claims, is \$59,240. Brown (1993) also reports other results, where the data have been adjusted for speed of finalization. In this case the required reserves are \$54,863. We also applied our Bayesian model without including any information on claims or on exposure. The result was \$58,592, very close to the chain-ladder. In addition we have the predictive distribution of the reserves.

Figure 2 shows the predictive distribution of the reserves generated with 5000 simulations of the Bayesian year-by-year model using the number of closed claims in each development-year/business-year cell. In this case the mean is \$50,058, while the median is \$49,818 and the standard deviation is \$4,607. When the model is used with the business year exposures (Section 4.2) then the mean is \$55,153, the median \$54,388 and the standard deviation is \$8,577. In this case the exposure is taken as the estimated ultimate number of claims. Figure 3 shows the corresponding distribution of the reserves. There is a clear difference between these two Bayesian 'estimates'. However, either one of these methods produces the distribution and hence is preferable to others since it yields much more information: we have the complete and exact distribution. In all cases the predicted reserves are smaller than the result obtained with the chain-ladder.

Table 5

Accident Year	Cumulative Closed Claims					Estimated Ultimate Number of Claims
	Development Year					
	0	1	2	3	4	
1988	400	700	850	930	1000	1000
1989	480	790	1,000	1,140		1,200
1990	500	950	1,190			1400
1991	570	1,050				1500
1992	600					1500

We have not estimated the required reserves in the Brown (1993) example using the global model. We will illustrate its application in the next example. We use this set of data to exemplify the application of the Bayesian approach when negative incremental claims are present. To do this we modify one cell in this data to look as Table 4a. When we applied the procedure as indicated in Section 4, ignoring the cell that has the negative increment (6,000) and the next one (15,600), the estimates of $(\underline{\hat{\theta}},\tilde{\sigma}^2)$ did not change significantly and the predictive mean of the reserves is again \$50,057.

Table 4a

Accident Year	Cumulative Loss Payments — Development Year 0	1	2	3	4	
1988	2,000	6,000	9,000	11,200	14,000	
1989	2,600	6,840	**6,000**	15,600		
1990	2,380	8,960	14,400			
1991	3,120	10,800				
1992	3,800					

Figure 1
Number of Claims, Brown Data (closed claims)

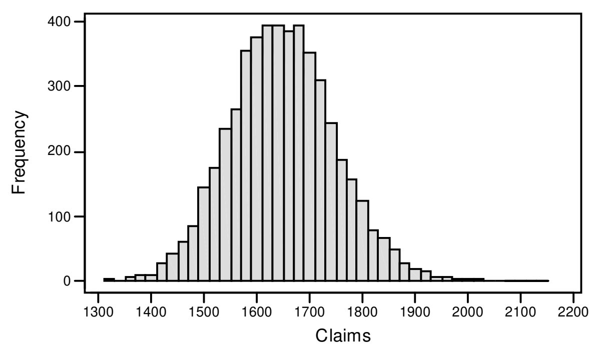
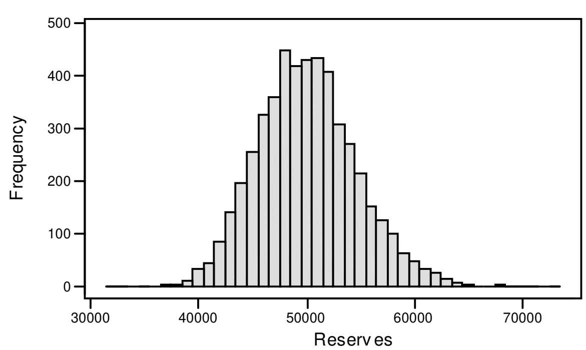
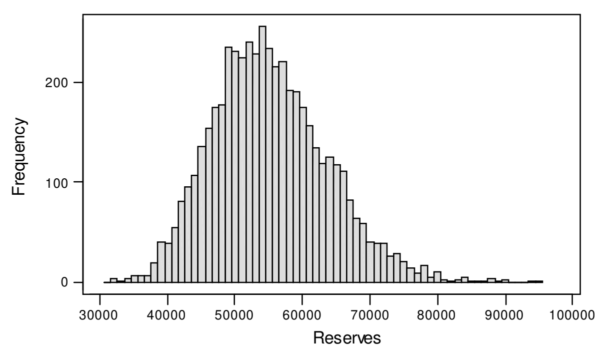


Figure 2
Reserves, Brown Data (closed claims)



We now apply the Bayesian methods to an example from Verrall (1990). Again, first we analyze the number of claims using the discrete model and then the claim amounts using the continuous model. Table 6 shows the claim amounts paid by development year for each year of origin. The number of claims is given in Table 7. Numbers in italics in the lower right triangle of this table are the estimates obtained by the chain-

Figure 3
Reserves, Brown Data (exposure)



ladder model with an estimate for the total of 1270, which compares to a mean of 1104 and a median of 1093 derived with the Bayesian model. The standard deviation is 116.

Table 6

General Insurance Amount of Claims										
Accident Year	Development Years									
	1	2	3	4	5	6	7	8	9	10
1	357848	766940	610542	482940	527326	574398	146342	139950	227229	67948
2	352118	884021	933894	1183289	445745	320996	527804	266172	425046	
3	290507	1001799	926219	1016654	750816	146923	495992	280405		
4	310608	1108250	776189	1562400	272482	352053	206286			
5	443160	693190	991983	769488	504851	470639				
6	396132	937085	847498	805037	705960					
7	440832	847631	1131398	1063296						
8	359480	1061648	1443370							
9	376686	986608								
10	344014									

Source: Verral(1990), Op.cit.

Figure 4 presents the results from 5000 simulations of this method, for the number of claims.

Figure 4
Number of Claims, Verrall Data (closed claims)

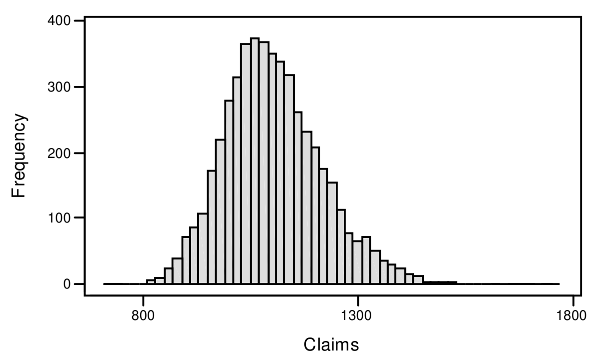
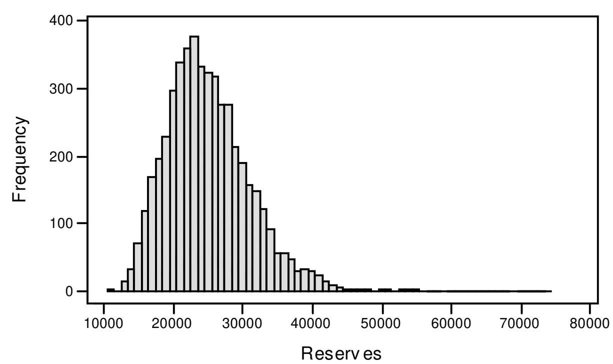


Table 7

Accident Year	General Insurance Number of Claims — Development Years 1	2	3	4	5	6	7	8	9	10	Outstanding Claims Chain Ladder
1	40	124	157	93	141	22	14	10	3	2	
2	37	186	130	239	61	26	23	6	6	*0*	*0*
3	35	158	243	153	48	26	14	5	*1*	*0*	*1*
4	41	155	218	100	67	17	6	*4*	*1*	*0*	*5*
5	30	187	166	120	55	13	*12*	*4*	*1*	*0*	*17*
6	33	121	204	87	37	*16*	*13*	*7*	*1*	*0*	*37*
7	32	115	146	103	*42*	*17*	*20*	*8*	*2*	*0*	*89*
8	43	111	83	*80*	*46*	*27*	*23*	*8*	*1*	*1*	*186*
9	17	92	*142*	*88*	*72*	*31*	*25*	*7*	*3*	*4*	*372*
10	22	*101*	*156*	*138*	*83*	*34*	*20*	*13*	*11*	*7*	*563*
											1270

Source: Verral(1990), Op.cit.

Figure 5
Reserves, Verrall Data (closed claims)



We now present results of applying all three Bayesian methods to obtain estimates of the required reserves. The regular chain-ladder method yields an estimate of 18,681. The upper portion of Table 8 shows the results of estimating the required reserves under the 'year-by-year' Bayesian model using a measure of exposures per accident year. This is the one that can be compared to the results of Verrall (1990), Mack (1991), Renshaw and Verrall (1994), and England and Verrall (1999) since all these are based on the use of some measure of business year exposures. Figure 5 shows the predictive density of the reserves using this model. The mean is 19,627 and the standard deviation 3,059. The results labeled as Renshaw and Christofides are taken directly from the last reference. We also include the Bayesian model using only the product of the posterior mean of the number of claims and the posterior median for the average claim amount, in each cell. It is labeled 'avge./cell' in the table. As mentioned before, with this approach the predictive mean of the number of claims is multiplied by the predictive mean (or median) of the average claim in each cell. It is the easiest to apply and the variance is not computed, since it requires estimating the variances and the second moments.

Taylor and Ashe used information on number of claims finalized ## (closed) in each period. Their estimate was 22,301. In the lower portion of Table 8 the predictive mean of the number of claims is also obtained using the information on 'closed' claims.

Table 8

Model	Reserves (1000)	Standard Deviation
Chain Ladder	18,681	-
Verral (no prior info)	19,512	3,194
Verral (prior info)	16,743	1,996
Mack	18,085	-
Renshaw/Christofedes	19,512	3,121
Bayesian (Avge./cell)	18,614	-
Bayesian (Year)	19,627	3,059
Taylor and Ashe	22,301	-
Bayesian (Avge./cell)	22,764	-
Bayesian (Year)	25,156	6,193
Bayesian (Global)	22,217	5,432

We applied the Simulation process to the Verrall (1990) data, but now using the information on closed claims in each development year for every accident year. The results are given in the lower portion of Table 8. In Figure 7 we show the distribution of the number of outstanding claims obtained using the 'global' model.

Figure 6
Reserves, Verrall Data (exposure)

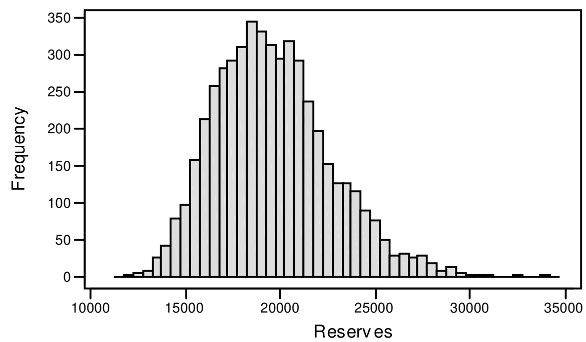
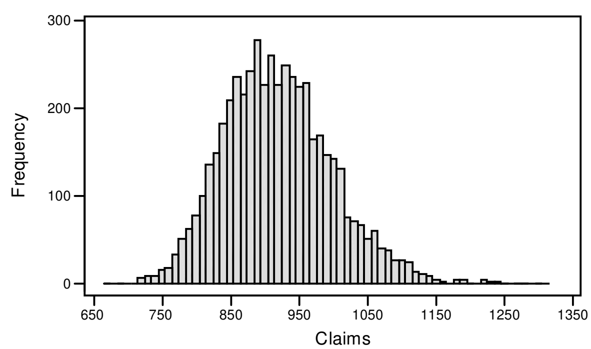
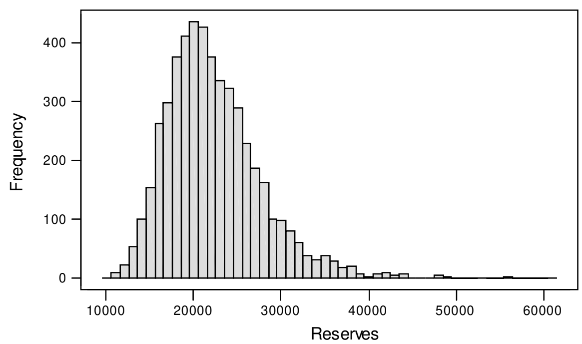


Figure 7
Number of Claims, Verrall Data (Global, closed claims)



From the simulation we obtained a mean of 919 for the number of outstanding claims, with a standard deviation of 79.51. These values compare with 1270 from the chain-ladder method and 1104 of the 'year-by-year' model. The resulting distribution for the total reserves is shown in Figure 8. The mean is 22,217 and the standard deviation is 5,432. These values are in line with the Bayesian results from the 'year-by-year' model and Doray (1996). However they are smaller than those obtained by Verrall (1990) or the chain-ladder. This may be due to the fact that the latter method utilizes only the aggregate claims data to estimate. It does not even use the exposures for each account year. Verrall (1990, 1991), on the other hand divides the exposures for each year of business into the claims data before the analysis is carried out. He uses more aggregated data than we do, since we are using the number of claims per year, as given in Taylor and Ashe (1983). He is modeling the claim amounts, but is taking the exposures as given. We are also modeling the number of claims finalized in each development year. In addition we have the complete distribution.

Figure 8
Reserves, Verrall Data (Global, closed claims)



7. FINAL REMARKS

From the example presented it is clear that the Bayesian procedure is more general than others. When the most aggregated data is used, then it yields values very close to those of the chain-ladder method. In fact, it can be considered as a Bayesian formalization of the chain-ladder method. The results show that the chain-ladder method and Verrall's approaches yield the lowest values for the reserves. This may be due to the fact that aggregate information is used. The Bayesian method modified to include more detailed information yields larger values for the mean and the variance, reflecting the uncertainty involved in estimating each cell.

Our method has the advantage of producing the predictive distribution of the required reserves, whereas with other models, even though you may have the variance, the complete distribution is not available, Renshaw and Verrall (1994) and England and Verrall (1999). This is also the case with Doray (1996), who computes UMVUE's for the mean and variance , but can not display the distribution. It is necessary in order to compute the probability of catastrophes, especially if use of the normal approximation is not warranted. In some of the cases presented here there is a clear skewness of the distribution, so that normality is not adequate. Confidence intervals obtained with Normal approximations can be very different from exact ones. The method has the advantage that it makes use of all the information available, both on the number and amounts of claims.

Our method also has the advantage that the number of claims is modeled as Poisson, while claim amounts are modeled using a log-normal distribution. In addition, whatever information is available on exposure, closed claims, etc. can be incorporated into the model without major changes. As well as negative incremental claims.

8. REFERENCES

(1) de Alba, E. (1988), Pronostico Bayesiano de Agregados en Procesos Estacionales Estables, *Revista de Estadistica*, Vol II(4), 1-6, INEGI, Mexico.

(2) de Alba, E. and M. Juárez (1995), Bayesian Forecasting: An Application to IBNR Claims Reserves, Proceedings of the Section on Bayesian Statistical Inference, ASA, 189-194.

(3) de Alba, E. and Mendoza, M. (1996), Discrete Bayesian Models for Forecasting with Stable Seasonal Patterns, *Advances in Econometrics*, Vol. 11, Part B., 267-281.

(4) Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A. and Nesbitt, C.J. (1986), *Actuarial Mathematics*, Society of Actuaries, Itasca, Illinois

(5) Brown, R. L. (1993), *Introduction to Ratemaking and Loss Reserving for Property and Casualty Insurance*, ACTEX Publications.

(6) Buhlmann, H., Schnieper, R. and Straub, E. (1980), Claims Reserves in Casualty Insurance Based on a Probabilistic Model, *Bulletin of the Association of Swiss Actuaries* 4(1), 21-45.

(7) Dickson, D.C.M., Tedesco, L.M. and Zenwirth, B. (1998), Predictive Aggregate Claims Distributions, *Journal of Risk and Insurance* 65, 689-709

(8) Doray, L.G. (1996), UMVUE of the IBNR reserve in a lognormal linear regression model, *Insurance: Mathematics and Economics 18*, 43-57, Elsevier Science B.V.

(9) England, P. and R. Verrall (1999), Analytic and bootstrap estimates of prediction errors in claims reserving, *Insurance: Mathematics and Economics*, Vol. 25, 281-293

(10) Ferguson, T.S. (1973), A Bayesian Analysis of Some Nonparametric Problems, *The Annals of Statistics* 1(2), 209-230.

(11) Goovaerts, M.J., et.al. (1990), *Effective Actuarial Methods*, North-Holland, Amsterdam

(12) Goovaerts, M.J. and H. Redant (1999), On the distribution of IBNR reserves, *Insurance: Mathematics and Economics*, Vol. 25, 1-9

(13) Haastrup, S. and E. Arjas (1996), Claims Reserving in Continuous Time; A Nonparametric Bayesian Approach, *ASTIN Bulletin* 26(2), 139-164.

(14) Hesselager, O. and Witting, T. (1988), A Credibility Model with Random Fluctuations in delay Pprobabilities for the Prediction of IBNR Claims, *ASTIN Bulletin* 18(1), 79-90.

(15) Hossack, I.B., Pollard, J.H. and Zenwirth, B. (1983), *Introductory Statistics with Applications in General Insurance*, U. Press, Cambridge

(16) Jewell, W.S. (1989), Predicting IBNYR Events and Delays. I Continuous Time, *ASTIN Bulletin* 19(1), 25-56.

(17) Jewell, W.S. (1990), Predicting IBNYR Events and Delays. II DiscreteTime, *ASTIN Bulletin* 20(1), 93-111

(18) Johnson, N.L. and Kotz, S. (1970), *Continuous Univariate Distributions*-1, Griffin, New York

(19) Karlin, S. (1968), *A First Course in Stochastic Processes*, Academic Press, New York.

(20) Kremer, (1982), IBNR claims and the two-way model of ANOVA, *Scandinavian Actuarial Journal*, 47-55

(21) Mack, T. (1994), Which Stochastic Model is Underlying the Chain Ladder Method?, *Insurance: Mathematics and Economics*, Vol. 15, 133-138.

(22) Norberg, R. (1986), A Contribution to Modelling of IBNR Claims, *Scandinavian Actuarial Journal*, 155-203

(23) Norberg, R. (1993), Prediction of Outstanding Liabilities in Non-Life Insurance, *ASTIN Bulletin* 23(1), 96-115.

(24) O'Hagan, A. (1994), *Kendall´s Advanced Theory of Statistics, Vol. 2B*, *Bayesian Statistics,* Edward Arnold

(25) O'Hagan, A. (1995), Fractional Bayes Factors for Model Comparison, *Journal of the Royal Statistical Society*, Series B, Vol 57,1, 99-138

(26) Oliver, R.M. (1987), Bayesian Forecasting with Stable Seasonal Patterns, *Journal of Business and Economic Statistics* 5, 77-86.

(27) Raiffa, H. and Schlaifer, R.(1961), *Applied Statistical Decision Theory*, Harvard, Boston.

(28) Renshaw, A.E. and R. Verrall (1994), A stochastic model underlying the chain-ladder technique, *Proceedings XXV ASTIN Colloquium*, Cannes.

(29) Taylor, G.C. (1986), *Claim Reserving in Non-Life Insurance,* Elsevier Science Publishers, New York

(30) Taylor, G.C. and F.R. Ashe (1983), Second Moments of Estimates of Outstanding Claims, *Journal of Econometrics* 23, 37-61.

(31) Verral, R. (1990), Bayes and Empirical Bayes Estimation for the Chain Ladder Model, *ASTIN Bulletin* 20(2), 217-243.

(32) Verral, R. (1991), On the estimation of reserves from loglinear models, *Insurance: Mathematics and Economics*, Vol. 10, 75-80

(33) Verral, R. (2000), An investigation into stochastic claims reserving models and the chain-ladder technique, *Insurance: Mathematics and Economics*, Vol. 26, 91-99

APPENDIX

We integrate equation (21) in the text with respect to $\underline{\theta}$ to get

$$
\begin{aligned}
f(\underline{p}, n_2, n_3, \ldots, n_k | D) \quad = \quad & \frac{n_1!}{\prod_{t=1}^{k} x_{1t}!} \prod_{t=1}^{k} p_t^{x_{1t}} \times \\
& \frac{n_2!}{x_{nk}!\, x^*_{2k-1}!} (1-p^*_{k-1})^{x_{2k}}\, p_{k-1}^{*x^*_{2k-1}} \frac{x^*_{2k-1}!}{\prod_{t=1}^{k-1} x_{2t}!} \prod_{t=1}^{k-1} \left(\frac{p_t}{p^*_{k-1}} \right)^{x_{2t}} \times \\
& \frac{n_3!}{(n_3 - x^*_{3k-2})!} (1-p^*_{k-2})^{x_3 - x^*_{3k-2}}\, p_{k-2}^{*x^*_{3k-2}} \frac{x^*_{3k-2}!}{\prod_{t=1}^{k-2} x_{3t}!} \prod_{t=1}^{k-2} \left(\frac{p_t}{p^*_{k-2}} \right)^{x_{3t}} \times \\
& \vdots \\
& \frac{n_k!}{(n_k - x_{k1})!\, x^*_{k1}!} (1-p^*_1)^{n_k - x_{k1}}\, p_1^{*x_{k1}} \frac{x^*_{k1}!}{x_{k1}!} \prod_{t=1}^{1} \left(\frac{p_1}{p^*_1} \right)^{x_{k1}} .
\end{aligned}
$$

We can now integrate out the N's and we are left with

$$
f(\underline{p} | D) \propto \left[\prod_{t=1}^{k} p_t^{x_{1t}} \right] \left[\prod_{t=1}^{k-1} \left(\frac{p_t}{p^*_{k-1}} \right)^{x_{2t}} \right] \left[\prod_{t=1}^{k-2} \left(\frac{p_t}{p^*_{k-2}} \right)^{x_{3t}} \right] \left[\prod_{t=1}^{k-3} \left(\frac{p_t}{p^*_{k-3}} \right)^{x_{4t}} \right] \cdots \left[\prod_{t=1}^{2} \left(\frac{p_t}{p^*_2} \right)^{x_{kt}} \right]
$$

where $p^*_s = (p_1 + p_2 + p_3 + \ldots + p_s)$. We will now show that we can write $f(\underline{p}|D)$ as

$$
\propto f(p_k | D) \cdot f(p_{k-1} | p_k, D) \cdot f(p_{k-2} | p_{k-1}, p_k, D) \ldots f(p_2 | p_3, p_4 \ldots p_k, D) f(p_1 | p_2 .. p_k, D)
$$

with $f(p_{k-i} | p_{k-i+1}, p_{k-i+2}, .. p_k, D) \propto \left(\frac{p_{k-i}}{p^*_{k-i}} \right)^{a_{k-i}-1} \left(1 - \frac{p_{k-i}}{p^*_{k-i}} \right)^{b_{k-i}-1}$, $a_{k-i} = \sum_{j=1}^{i+1} x_{jk-i} + 1$ and $b_{k-i} = \sum_{t=1}^{k-i-1} \sum_{j=1}^{i+1} x_{jt} + 1$.

To do this notice that

$$f(\underline{p} \mid D) \propto \underbrace{\left(\prod_{t=1}^{k} p_t^{x_{it}}\right)}_{A_1} \underbrace{\left[\prod_{t=1}^{k-1}\left(\frac{p_t}{p_{k-1}^*}\right)^{x_{2t}}\right]}_{A_2} \underbrace{\left[\prod_{t=1}^{k-2}\left(\frac{p_t}{p_{k-2}^*}\right)^{x_{3t}}\right]}_{A_3} \times \ldots \times \underbrace{\left[\prod_{t=1}^{2}\left(\frac{p_t}{p_2^*}\right)\right]^{x_{k-1,t}}}_{A_{k-1}} \qquad \text{(A.1)}$$

using: $p_{k-1}^* = 1 - p_k$, $p_{k-2}^* = 1 - p_{k-1} - p_k$ and, in general, $p_s^* = p_1 + p_2 + .. + p_s = 1 - p_{s+1} - ... - p_k$. Now rewrite the term A_1 in the expression above as follows

$$A_1 = p_k^{x_{1k}} \prod_{t=1}^{k-1} p_t^{x_{1t}} = p_k^{x_{1k}} \prod_{t=1}^{k-1}\left(\frac{p_t}{p_{k-1}^*}\right)^{x_{1t}} \prod_{t=1}^{k-1}\left(p_{k-1}^*\right)^{x_{1t}} = p_k^{x_{1k}} (1-p_k)^{\sum_{t=1}^{k-1} x_{1t}} \prod_{t=1}^{k-1}\left(\frac{p_t}{p_{k-1}^*}\right)^{x_{1t}}$$

$$= p_k^{x_{1k}} (1-p_k)^{\sum_{t=1}^{k-1} x_{1t}} \prod_{t=1}^{k-1}\left(\frac{p_t}{p_{k-1}^*}\right)^{x_{1t}}.$$

So that now

$$A_1 \times A_2 = \underbrace{\left[p_k^{x_{1k}} (1-p_k)^{\sum_{t=1}^{k-1} x_{1t}}\right]}_{B_1} \left[\prod_{t=1}^{k-1}\left(\frac{p_t}{p_{k-1}^*}\right)^{x_{1t}+x_{2t}}\right]$$

$$= B_1 \left(\frac{p_{k-1}}{p_{k-1}^*}\right)^{x_{1k-1}+x_{2k-1}} \left(1-\frac{p_{k-1}}{p_{k-1}^*}\right)^{\sum_{t=1}^{k-2}(x_{1t}+x_{2t})} \prod_{t=1}^{k-2}\left(\frac{p_t}{p_{k-1}^*}\right)^{x_{1t}+x_{2t}} \left(1-\frac{p_{k-1}}{p_{k-1}^*}\right)^{-\sum^{k-2}(x_{1t}+x_{2t})}$$

$$= \underbrace{B_1 \left(\frac{p_{k-1}}{p_{k-1}^*}\right)^{x_{1k-1}+x_{2k-1}} \left(1-\frac{p_{k-1}}{p_{k-1}^*}\right)^{\sum_{t=1}^{k-2}(x_{1t}+x_{2t})}}_{B_2} \prod_{t=1}^{k-2}\left(\frac{p_t}{p_{k-1}^*}\right)^{x_{1t}+x_{2t}} \left(\frac{p_{k-1}^* - p_{k-1}}{p_{k-1}^*}\right)^{-\sum^{k-2}(x_t+x_{2t})}$$

$$= B_1 \times B_2 \times \prod_{t=1}^{k-2}\left(\frac{p_t}{p_{k-1}^*}\right)^{x_{1t}+x_{2t}} (p_{k-2}^*)^{-\sum_{t=1}^{k-2}(x_{it}+x_{2t})} (p_{k-1}^*)^{\sum^{k-2}(x_{1t}+x_{2t})},$$

since $p_{k-1}^* - p_{k-1} = 1 - p_k - p_{k-1} = p_{k-2}^*$. Furthermore

$$A_1 \times A_2 = B_1 \times B_2 \times \prod_{t=1}^{k-2}\left(\frac{p_t}{p_{k-2}^*}\right)^{x_{1t}+x_{2t}}.$$

We proceed in a similar fashion to find

$$A_1 \times A_2 \times A_3 = B_1 \times B_2 \times \prod_{t=1}^{k-2}\left(\frac{p_t}{p_{k-2}^*}\right)^{x_{1t}+x_{2t}} \times \left[\prod_{t=1}^{k-2}\left(\frac{p_t}{p_{k-2}^*}\right)^{x_{3t}}\right]$$

$$= B_1 \times B_2 \times \underbrace{\left(\frac{p_{k-2}}{p_{k-2}^*}\right)^{x_{1k-2}+x_{2k-2}+x_{3k-2}}\left(1-\frac{p_{k-2}}{p_{k-2}^*}\right)^{\sum_{t=1}^{k-3}(x_{1t}+x_{2t}+x_{3t})}}_{B_3} \times$$

$$\left(1-\frac{p_{k-2}}{p_{k-2}^*}\right)^{-\sum_{t=1}^{k-3}(x_{1t}+x_{2t}+x_{3t})} \prod_{t=1}^{k-3}\left(\frac{p_t}{p_{k-2}^*}\right)^{x_{1t}+x_{2t}+x_{3t}}$$

so

$$A_1 \times A_2 \times A_3 = B_1 \times B_2 \times B_3 \times \prod_{t=1}^{k-3}\left(\frac{p_t}{p_{k-3}^*}\right)^{x_{1t}+x_{2t}+x_{3t}},$$

and so on, until we have $A_1 \times A_2 \times ... \times A_{k-1} = B_1 \times B_2 \times ... \times B_{k-1} \times B_k$, with

$$B_i = \left(\frac{p_{k-i+1}}{p_{k-i+1}^*}\right)^{x_{1l-i+1}+x_{2k-i+1}+...+x_{ik-i+1}}\left(1-\frac{p_{k-i+1}}{p_{k-i+1}^*}\right)^{\sum_{t=1}^{k-i}(x_{1t}+x_{2t}+...+x_{it})} \tag{A.2}$$

where $B_k = 1$ and $B_i = f(p_{k-i+1} \mid p_{k-i+2}, p_{k-i+3}, ..., p_k, D)$, as required. Recall $x_i^* = \sum_{t=1}^{a_i} x_{it}$, $p_{a_i}^* = p_1 + .. + p_{a_i}$, $p_{k-1}^* = 1 - p_k$. Let $p_{k-2}^* = 1 - p_{k-1} - p_k = 1 - \tilde{p}_{k-1}$ with $\tilde{p}_{k-1} = p_{k-1} + p_k$ and $\tilde{p}_i = p_i + p_{i+1} + .. + p_k = 1 - p_{i-1}^*$. Using these definitions it follows immediately that

$$E(p_k \mid D) = \frac{x_{1k}+1}{\sum_{t=1}^{k} x_{1t} + 2} = \hat{p}_k,$$

$$E(p_{k-1} \mid p_k, D) = (1-p_k)\left(\frac{x_{1k-1} + x_{2k-1} + 1}{\sum_{t=1}^{k-1}(x_{1t} + x_{2t}) + 2}\right) = (1-p_k)\ \hat{p}_{k-1},$$

$$E(p_{k-2} \mid p_{k-1}, p_k, D) = (1-p_k)(1-p_{k-1}) \times \left(\frac{\sum_{i=1}^{3} x_{ik-2} + 1}{\sum_{t=1}^{k-2}\sum_{i=1}^{3} x_{it} + 2}\right) = (1-p_k)(1-p_{k-1})\ \hat{p}_{k-2},$$

and so on. The general unconditional expected value is then given by:

$$E(p_{k-i} \mid D) = (1-\hat{p}_k)(1-\hat{p}_{k-1}) \cdot \ldots \cdot (1-\hat{p}_{k-i+1})\ \hat{p}_{k-i}.$$

The conditional posterior pdf's given in (A.2) are essentially the same as the 'predictive' distributions given in Verral (2000) [his equation (3.11)]. The unconditional means given in (A.3) is also very similar to his results [equation (4.1)]. In our case, the variances are much more complicated to compute due to the successive conditioning that has to be carried out, and to the fact that we use more detail. To find the mean of outstanding claims (OC) in accident year 2 we have:

$$E(OC_2 \mid D) = E(p_k N_2 \mid D) = \underset{p_k}{E}\left[E(N_2 \mid p_k)\, p_k\right] = \underset{p_k}{E}\left[p_k\ \frac{x^*_{2k-1}}{p^*_{k-1}}\right],$$

and from Raiffa and Schlaiffer (1961, page 215):

$$= \underset{p_k}{E}\left[p_k \cdot \frac{\sum_{t=1}^{k-1} x_{2t}}{1-p_k} \mid D_t\right] = \sum_{t=}^{k-1} x_{2t}\ \underset{p_k}{E}\left[\frac{p_k}{1-p_k} \mid D_t\right] = \sum_{t=1}^{k-1} x_{2t} \cdot \left(\frac{x_{1k} + 1}{\sum_{t=1}^{k-1} x_{1t}}\right) \cong x_{1k}\left(\frac{\sum_{t=}^{k-1} x_{2t}}{\sum^{k-1} x_{1t}}\right).$$

Similarly

$$E(OC_3 \mid D) = E(\tilde{p}_{k-1} N_3 \mid D) = \underset{p_{k-1}}{E}\left[E(N_3 \mid \underline{p})(p_k + p_{k-1})\right] =$$

$$= \underset{p}{E}\left[(p_k + p_{k-1})\left(\sum_{t=1}^{k-2} \frac{x_{3t}}{p^*_{k-2}}\right) \mid D\right] = \underset{p}{E}\left[\frac{(1-p^*_{k-2})}{p^*_{k-2}} \sum_{t=1}^{k-2} x_{3t} \mid D\right] = \qquad \text{(A.3)}$$

$$\cong \left[(x_{1k} + x_{1k-1}) + \underbrace{\left(x_{1k} \frac{\sum^{k-1} x_{2t}}{\sum^{k-1} x_{1t}} \right)}_{E(R_2|D)=x_{2k}} + x_{2k-1} \right] \left(\frac{\sum^{k-2} x_{3t}}{\sum^{k-2} (x_{1t} + x_{2t})} \right).$$

In general

$$E(OC_i \mid D) \cong \left[\sum^{k-(i-1)} x_{it} \Big/ \left(\sum_{t=1}^{k-(i-1)} (x_{1t} + x_{2t} + .. + x_{(i-1)t}) \right) \right] \times \left[\sum_{j=1}^{i-1} \sum_{t=k-(i-2)}^{k*} x_{jt} \right],$$

where $x_{jt} = E(OC_j \mid D)$ whenever $j + t > k + 1$.