A Bayesian nonparametric dynamic AR model for multiple time series analysis

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Outline

1. Model
2. Dependent Pólya trees
3. Analysis
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Destationalized Series ITAEE 2008
The economy of the 32 Mexican states depends on the global economy of the whole country.

The 32 time series interact among each other.

Need a model that respects the individual evolution of each time series but considers the country dependence.
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Let $X_i = \{X_{ti}, t \geq 1\}$, $i = 1, \ldots, n$. Propose

$$X_{ti} = \beta_1 X_{t-1,i} + \cdots + \beta_p X_{t-p,i} + \varepsilon_{ti},$$

with $X_{ti} = 0$ w.p.1 for $t < 0$, and

$$\varepsilon_{ti} \sim F_t, \quad \text{for} \quad i = 1, \ldots, n$$

$$\{F_1, F_2, \ldots\} \sim \text{dPT}_q(\Pi, a, \rho, C)$$

$$\theta \sim f(\theta).$$
Pólya tree

\[ Y_0 = p(B_0 | B) \]

\[ Y_1 = p(B_1 | B) = 1 - Y_0 \]
Pólya tree
Formally, a random probability measure $F$ in $(\mathbb{R}, \mathcal{B})$ has a Pólya tree distribution with parameters $(\Pi, \mathcal{A})$.

In notation $F \sim \text{PT}(\Pi, \mathcal{A})$, if there exists a sequence of non-negative numbers $\mathcal{A} = \{\alpha_{mj}\}$ and a family of r.v. $\mathcal{Y} = \{Y_{mj}\}$ s.t.

a) All r.v. in $\mathcal{Y}$ are independent;

b) For each $(m, j), j = 1, \ldots, 2^{m-1}$ and $m = 1, 2, \ldots$, $Y_{m,2j-1} \sim \text{Be}(\alpha_{m,2j-1}, \alpha_{m,2j})$ and $Y_{m,2j} = 1 - Y_{m,2j-1}$; and,

c) For each $m = 1, 2, \ldots$ and each $j = 1, \ldots, 2^m$,

$$F(B_{mj}) = \prod_{k=1}^{m} Y_{m-k+1,j_{m-k+1}^{(m,j)}},$$

where $j_{k-1}^{(m,j)} = \lceil j_{k}^{(m,j)}/2 \rceil$ is a recursive formula with initial value $j_{m}^{(m,j)} = j$.

Typically $\alpha_{m,j} = a \rho(m)$ \Rightarrow $F \sim \text{PT}(\Pi, a, \rho)$.
Pólya tree

- Randomness in the tree depends on $Y_{m+1,2j-1} = P(B_{m+1,2j-1} | B_{mj})$

- Introduce dependence across several trees by defining a sequence of dependent variables $\mathcal{Y}_t = \{Y_{t,m,j}\}$

- How?
Pólya tree

- Randomness in the tree depends on $Y_{m+1,2j-1} = P(B_{m+1,2j-1} | B_{mj})$

- Introduce dependence across several trees by defining a sequence of dependent variables $\mathcal{Y}_t = \{Y_{t,m,j}\}$

- How? Through a beta process (Jara et al., 2013)

$$y_t \mid u_t, u_{t-1}, \ldots, u_{t-q} \sim \text{Be} \left( a + \sum_{j=0}^{q} u_{t-j}, b + \sum_{j=0}^{q} (c_{t-j} - u_{t-j}) \right) ,$$

$$u_t \mid w \sim \text{Bin}(c_t, w), \quad t = 1, 2, \ldots$$

$$w \sim \text{Be}(a, b)$$
Order $q$ beta process ($\text{BeP}_q$)
Dependent Pólya trees

- \( \mathcal{F} = \{F_1, F_2, \ldots\} \) are dependent Pólya trees s.t.
  \[
  \mathcal{F} \sim d\text{PT}_q(\Pi, a, \rho, \mathcal{C}),
  \]
  with \( \mathcal{C} = \{c_{t,m,j}\} \), \( a > 0 \) and \( \rho(m) = m^\delta \), \( \delta > 1 \) to ensure continuity. Usually \( \delta = 2 \) but we suggest \( \delta = 1.1 \) (Watson et al., 2017)

- Properties:
  \[
  \text{Corr}\{F_t(B_{mj}), F_{t+s}(B_{mj})\} = \frac{\prod_{k=1}^{m} \left\{ \psi_{t,s,m-k+1,j_{m-k+1}^{(m,j)}(m,j)} \sigma_{m-k+1}^2 + 1/4 \right\} - (1/4)^m}{\prod_{k=1}^{m} \left\{ \sigma_{m-k+1}^2 + 1/4 \right\} - (1/4)^m},
  \]
  
  with
  \[
  \psi_{t,s,k,j_{k}^{(m,j)}(m,j)} = \frac{2a\rho(k) \left( \sum_{l=0}^{q-s} c_{t-l,k,j_{k}^{(m,j)}} + \sum_{l=0}^{q} c_{t-l,k,j_{k}^{(m,j)}} \right) \left( \sum_{l=0}^{q} c_{t+s-l,k,j_{k}^{(m,j)}} \right)}{2a\rho(k) + \sum_{l=0}^{q} c_{t-l,k,j_{k}^{(m,j)}} \left( 2a\rho(k) + \sum_{l=0}^{q} c_{t+s-l,k,j_{k}^{(m,j)}} \right)},
  \]
  \[
  \sigma_k^2 = \frac{1}{4\{2a\rho(k) + 1\}}.
  \]
Correlation in dPT
Mixtures in dependent Pólya trees

- It is well known that PT have discontinuities at the partition boundaries.
- Generally \( \Pi = \{ B_{mj} \} \) is defined via the quantiles of \( F_0 \)

\[
B_{mj} = \left( F_0^{-1} \left( \frac{j-1}{2^m} \right), F_0^{-1} \left( \frac{j}{2^m} \right) \right)
\]

- It is possible to diminish the partition effect if we mix with respect to a parameter \( \theta \), i.e. \( \Pi_\theta = \{ B_{mj}^\theta \} \) defined by \( F_0(\cdot \mid \theta) \) with \( \theta \sim f(\theta) \) which implies

\[
\mathcal{F} \sim \int \text{dPT}_q(\Pi_\theta, a, \rho, C)f(\theta)d\theta
\]
Recall our model

\[ X_{ti} = \beta_1 X_{t-1,i} + \cdots + \beta_p X_{t-p,i} + \epsilon_{ti}, \]

with \( X_{ti} = 0 \) w.p.1 for \( t < 0 \), and

\[ \epsilon_{ti} \mid F_t \overset{iid}{\sim} F_t, \quad \text{for} \quad i = 1, \ldots, n \]

\[ \{F_1, F_2, \ldots\} \mid \theta \sim \text{dPT}_q(\prod_\theta, a, \rho, C) \]

\[ \theta \sim f(\theta). \]

Specifications:

- \( F_0(\cdot \mid \theta) = N(0, \theta^2) \)
- Fix median at zero: Take \( B_{11} = (\infty, 0] \) and \( B_{12} = (0, \infty) \) with \( F_t(B_{11}) = F_t(B_{12}) = 1/2 \iff Y_{t,1,1} = Y_{t,1,2} = 1/2 \) w.p.1.
Bayesian inference

Priors

- For $C$:
  
  $c_{t,m,2j-1} \mid \lambda_{m,2j-1} \overset{\text{ind}}{\sim} \text{Po}(\lambda_{m,2j-1})$, \quad $\lambda_{m,2j-1} \overset{\text{iid}}{\sim} \text{Ga}(\lambda_{m,2j-1})$

  for $t = 1, \ldots, T$, $m = 1, 2, \ldots$ and $j = 1, \ldots, 2^{m-1}$

- For $\theta$:
  
  $\theta \sim \text{Ga}^{-1/2}(b_1^\theta, b_2^\theta)$

- For the AR coefficients $\beta$:
  
  $\beta_{ki} \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_\beta^2)$,

  for $k = 1, \ldots, p$ and $i = 1, \ldots, n$
Data analysis

- Let $Y_{ti}$ be the ITAEE observation for state $i$ at time $t$
- Remove level, tendency and seasonality via second differences, i.e.
  \[ X_t = (Y_t - Y_{t-1}) - (Y_{t-1} - Y_{t-2}), \text{ for } t = 3, \ldots, 46 \]
- Plots of the partial autocorrelation of $X_{ti}$ suggest an autoregressive dependence of order between 2 and 4
- Prior specifications : $(b_1^{\lambda}, b_2^{\lambda}) = (1, 1); (b_1^{\theta}, b_2^{\theta}) = (0.1, 0.1); \sigma_\beta^2 = 100$
- Take a finite tree with $M = 5$ levels, $a = 1$, and $\rho(m) = m^\delta$ with $\delta = 1.1$
- The values $(p, q)$ are determined via the DIC
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Second differences

![Graph showing second differences with time on the x-axis and density on the y-axis.](image-url)
Estimated $c_{t,m,j}$ parameters

**Figure:** $m = 2$ and $j = 3$ (solid line); $m = 3$ and $j = 5$ (dashed line); and $m = 4$ and $j = 9$ (dotted-dashed line)
Estimated error distribution
Quantiles of the error distributions
\[ P(\beta_{ki} > 0 \mid \text{data}) \]
References

